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BUSINESS MATHEMATICS - II

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Content Writer: *Dr Jagbir Singh*

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BUSINESS MATHEMATICS-II

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Unit I

Linear Programming-Formulation of LPP: Graphical method of solution; Problems relating to two variables including the case of mixed constraints; Cases having no solution, multiple solutions, unbounded solution and redundant constraints.

Unit-II

Simplex Method—Solution of problems up to three variables, including cases of mixed constraints; Duality; Transportation Problem.

Unit-III

Compound Interest: Certain different types of interest rates; Concept of present value and amount of a sum

Unit-IV

Annuities: Types of annuities; Present value and amount of an annuity, including the case of continuous compounding; Valuation of simple loans and debentures; Problems relation to sinking funds.

Suggested Readings:

- Allen B.G.D: Basic Mathematics; Mcmillan, New Delhi.
- Volra. N. D. Quantitative Techniques in Management, Tata McGraw Hill, New Delhi. Kapoor V.K. Business
- Mathematics: Sultan chand and sons, Delhi.

Contents

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Linear Programming

Structure

1.1. Introduction. For manufacturing a product, a number of resources like raw material, machines, manpower and other types of materials. These resources are available in limited quantity but their demand is more. So they are used be optimally utilized so that cost could be minimized and profits could be maximized. Linear programming is a technique which helps us in the optimal utilization of these resources.

1.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Formulation of LPP.

(ii) Graphical method of solution.

(iii) Cases having no solution, multiple solutions, unbounded solution and redundant constraints.

1.1.2. Keywords. Linear Programming, Unique Solution, Unbounded Solution.

1.2. Linear Programming. A linear programming problem consists of a linear function to be maximized or minimized subject to certain constraints in the form of linear equations or inequalities.

1.2.1. Characteristics of Linear Programming

1. Every linear programming problem has an objective which should be clearly identifiable and measurable. For example objective can be maximization of sales, profits and minimization of costs and so on.
2. All the products and resources should also be clearly identifiable and measurable.
3. Resources are available in limited quantity.
4. The relationship representing objectives and resources limitation are represented by constraint in equalities or equations. These relationships are linear in nature.

The above characteristics will be clear from the following example:

1.2.2. Example. A firm is engaged in manufacturing two products A and B. Each unit of A requires 2 kg. of raw material and 4 labour hours for processing while each unit of B requires 3 kg. of raw material and 3 labour hours. The weekly availability of raw material and labour hours is limited to 60 kg. and 96 hours respectively. One unit of product A sold for Rs. 40 while are unit of B is sold for Rs. 35.

In the above problem, first we define the objective. Since we are given data on sales prince per unit of two products, so our objective is to maximize the soles.

Let x_1 be the number of units of A and x_2 be the number of units of B to be produced. So Sales revenue from sale of x_1 unit of A = $40x_1$ and sales revenue from sale of x_2 units of B = $35x_2$. Thus

$$\text{Total sales} = 40x_1 + 35x_2 \text{ and}$$

Our objective becomes

$$\text{Maximise } Z = 40x_1 + 35x_2$$

After this we come to the constraint in equalities, from the given date, we find that quantity of raw material required to produce x_1 kg of A = $2x_1$ [each unit of A requires 2 kg of raw material so x_1 units of A require $2x_1$ kg of raw material]

In the same way, quantity of raw material required to produce x_2 units of B = $3x_2$.

So new total raw material requirement = $2x_1 + 3x_2$

But here we have a constraint in the form of quantity of raw material available. Since the maximum quantity of raw material available is limited to 60 kg, so we cannot use more than 60 kg of raw material in any case. Mathematically we can write it as

$$2x_1 + 3x_2 \leq 60$$

In other words we can say that total quantity of raw material consumed is less than and equal to (shown by the symbol $\leq$) 60

Similarly we can produced to expressed the labour constraints in the following way:

Labour hours required to produce to produce x_1 units of A = $3x_1$ and

Labour hours required x_2 units of B = $3x_2$.

Since the total number of labour hours available per week is limited to 96, we can express this constraint as

$$3x_1 + 3x_2 \leq 96$$

In the last we express the non-negativity condition, i.e. $x_1, x_2 \geq 0$

Which is self-clear because number of units of product A or B can be zero or positive only.

Now the above problem can be summarised as

$$\text{Max. } Z = 40x_1 + 35x_2$$

Subject to

$$2x_1 + 3x_2 \leq 60$$

$$3x_1 + 3x_2 \leq 96$$

$$x_1, x_2 \geq 0.$$

Generally we can express a linear programming in the following way:

$$\text{Maximise or Minimise } Z = C_1x_1 + C_2x_2 + \ldots + C_nx_n \quad \text{objective function}$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \text{ or } \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \text{ or } \geq b_2$$

$$\ldots \ldots \ldots \ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_n \text{ or } \geq b_n$$

$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

where C's are the profit or cost coefficients of decisions variables x's, a's are the resource coefficients and b's are the resource values.

1.3. Solutions of Linear Programming Problems.

There are two methods of solving the linear programming problems - Graphical method and Simplex method.

1.3.1. Graphical Method.

By this method, we can solve problems involving two variables only, one variable is taken as x and the second as y. On any graph we can show only two variables. The steps involved in this method are following:

1. We draw a horizontal straight line on a graph paper which is called x-axis.
2. We draw a vertical straight line which is perpendicular to this horizontal line. This perpendicular line is called Y-axis.
3. Constraints inequalities are converted into equations and plot their points on the graph.
4. Common feasible area of all the inequalities is found out. From this area, we get the maximum or minimum values of Z.

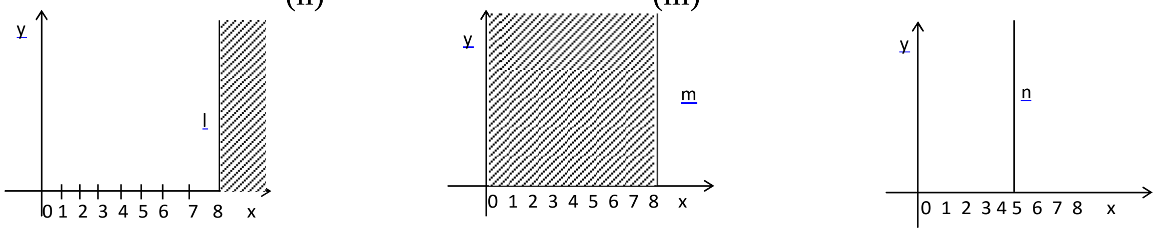
The following illustrations will make it more clear:

1.3.2. Example. Find the feasible area of the following :

(i) $x \leq 8$ (ii) $x \geq 6$ (iii) $x = 5$

Solution.

(i) (ii) (iii)



(Shaded portion shows the feasible area)

In the first case, feasible area is to the right of the vertical line *l*. There is no limit to its maximum value. In the second case, the feasible area is to the left of the line m forwards y-axis and in third case, feasible area is along the line n.

To find feasible area of an equation having two variables only.

1.3.3. Example. Find the feasible area of the following :

(i) $2x + y \leq 6$ (ii) $2x + y \leq 6$, $x, y \geq 0$.

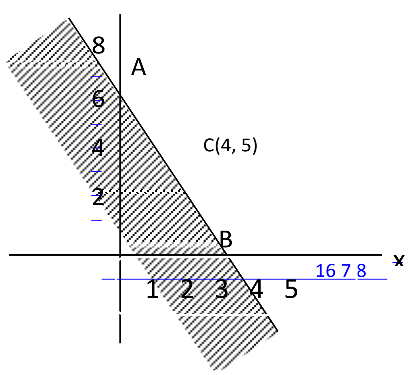
Solution (i) For plotting the lines, change the inequalities into equations so $2x+y=6$

Find two points to be plotted on the graph

When $x = 0$ $y = 6$

When $y = 0$ $x = 3$

So we get two points (0, 6) and (3, 0) which when plotted on the graph and joined by a line gives us the line of the above equation.

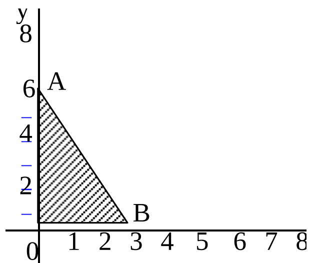


Now since there is no mention of the signs of x and y. So feasible area extends infinitely to the left of line joining points. A(0,6) and B(3, 0). Any point taken to the right of line joining A and B does not satisfy this condition of inequality. For example, let us consider point C(4, 5) Substituting the values of x = 4 and y = 5 in the L.H.S. of the inequality we get

$$2x + y = 2*4 + 5 = 13$$

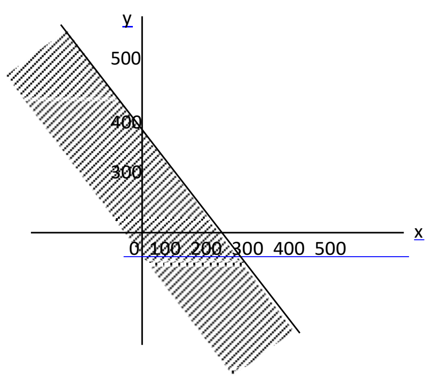
which is not less than 6. Hence this point does not lies in the feasible area of the in equation.

(ii) In this case, since we are given x, y ≥ 0, it means that neither x nor y can have negative sign. So the feasible area of the inequality 2x + y ≤ 6 lies in the region OAB.



1.3.4. Example. Draw the graph of the in equation $\frac{x}{200} + \frac{y}{300} \leq 1$. Which of the following points lie in the graph (i) 300, 0 (ii) 200, 400 (iii) 150, 250

Solution. Given $\frac{x}{200} + \frac{y}{300} \leq 1$



or $\quad 3x + 2y \leq 600$

Changing into equation

$$3x + 2y = 600$$

When x = 0, y = 300

y = 0, x = 200

From the graph, it is clear that

(i) Point (300, 0) does not lie in the graph.

(ii) Point (200, 400) also does not lie in graph but

(iii) Point (150, 250) lies in the graph.

13.5. Example. Draw the diagram of solution set of the linear constraints

$$2x + 3y \leq 6$$

$$x + 4y \leq 4$$

$$x \geq 0,\ y \geq 0.$$

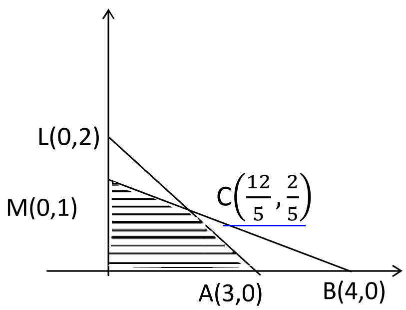
Solution. The given constraints are

$$2x + 3y \leq 6$$

$$x + 4y \leq 4$$

$$x \geq 0,\ y \geq 0.$$

Consider a set of rectangular cartesian axes OXY in the plane. Each point has co-ordinates of the type (x, y) and conversely. It is clear that any point which satisfies $x \geq 0$, $y \geq 0$ lies in the first quadrant.



Let us draw the graph of $2x + 3y = 6$

For $x = 0$, $3y = 6$, i.e., $y = 2$

For $y = 0$, $2x = 6$ i.e., $x = 3$

Therefore, line $2x + 3y = 6$ meets OX in A(3, 0) and OY in L(0, 2)

Again let us draw the graph of

$$x + 4y = 4$$

For $x = 0$, $4y = 4$ i.e., $y = 1$

For $y = 0$, $x = 4$

Therefore, line $x + 4y = 4$ meets OX in B(4, 0) and OY in M(0, 1).

Since feasible region is the region which satisfies all the constraints, feasible region is the quadrilateral OACM.

The corner points are O(0, 0), A(3, 0), C(12/5, 2/5), M(0, 1).

Note

1. Students must note that graphs are to be drawn on the graph paper.

2. Point C can also be calculated by solving the equation $2x + 3y = 6$, $x + 4y = 4$. It helps them in verifying the result obtained from the graph.

3. $2x + 3y \leq 6$ represents the region on and below the line AL. Similarly $2x + 3y \geq 6$ will represent the region on and above the line AL.

4. In order to have a clear picture of the region below or above the line, it is better to note that if the point (x_1, y_1) satisfies the in equation then the region containing this point is the required region.

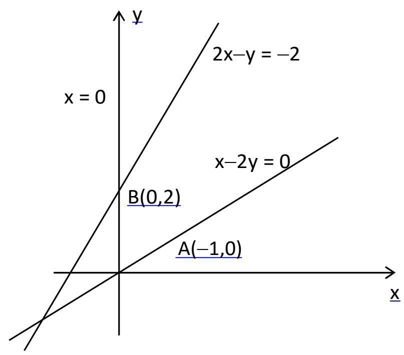
1.3.6. Example. Verify that the solution set of the following linear constraints is empty:

$$x - 2y \geq 0,\ 2x - y \leq -2,\ x \geq 0,\ y \geq 0.$$

Solution. The straight line $x - 2y = 0$ passes through O.

The straight line $2x - y = -2$ meets x-axis at A(-1, 0) and y-axis at B(0, 2).

We have the following figure.



Since no portion satisfies all the four constraints,

Therefore, the solution set is empty.

1.4. Solution of LPP by graphical method.

1.4.1. Maximisation case.

1.4.2. Example. A furniture dealer deals in only two items : tables and chairs. He has Rs. 5000.00 to invest and a space to store at most 60 pieces. A table costs him Rs. 250.00 and a chair Rs. 50.00. He can sell a table at a profit of Rs. 50.00 and a chair at a profit of Rs. 15.00. Assuming that he can sell all the items that he buys, how should he invest his money in order that he may maximize his profit ?

Solution. We formulate the problem mathematically.

Max. possible investment = Rs. 5000.00

Max. storage space = 60 pieces of furniture

	Cost	Profit
Table :	Rs. 250.00	Rs. 50.00
Cost Chair :	Rs. 50.00	Rs. 15.00

Let x and y be the number of tables and chairs respectively. Then we have the following constraints :

$x \geq 0$ …(1) $y \geq 0$ …(2)

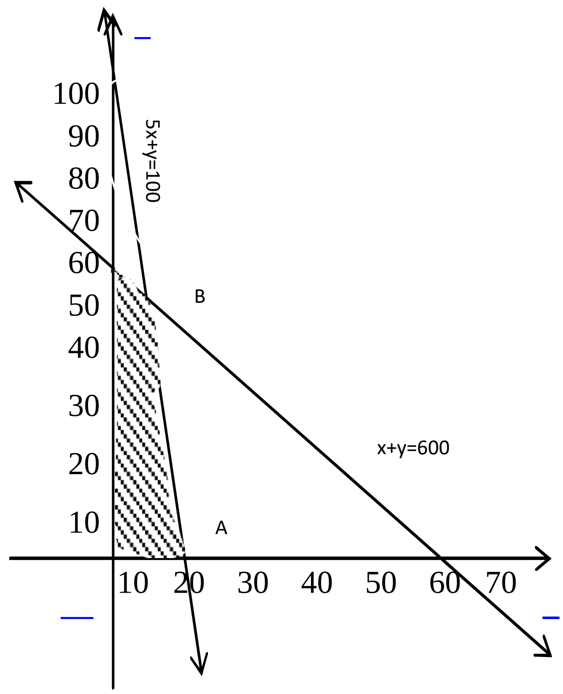
$250x + 50y \leq 5000$ i.e. $5x + y \leq 100$ …(3)

and $x + y \leq 60$ …(4)

Let Z be the profit, then $Z = 50x + 15y$ …(5)

We are to maximize Z subject to constraint (1), (2) (3) and (4).

Let us graph the constraints given in (1), (2), (3) and (4).



Explanation. Draw the straight lines x = 0, (y-axis), y = 0 (x-axis).

Draw the straight line x + y = 60. This meets the x-axis at (60, 0) and y-axis at (0, 60).

Draw the straight line 5x + y = 100. This meets x-axis at (20, 0) and y-axis at (0, 100).

The shaded region consists of points, which are the intersections of four constraints. This region is called feasible solution of the linear programming problem.

The vertices of the figure OABC show the possible combinations of x and y one of which gives us the maximum value. Now we consider the points one by one.

1.4.3. Example. A company manufactures two types of telephone sets, one of which is cordless. The cord type telephone set requires 2 hours to make, and the cordless model requires 4 hours. The company has at most 800 working hours per day to manufacture these models and the packing department can pack at the most 300 telephone sets per day. If the company sells, the cord type model for Rs. 300 and the cordless model for Rs. 400, how many telephone sets of each type should it produce per day to maximize its sales ?

Solution. Let the number of cord type telephone sets be x and the number of cordless type telephone sets be y.

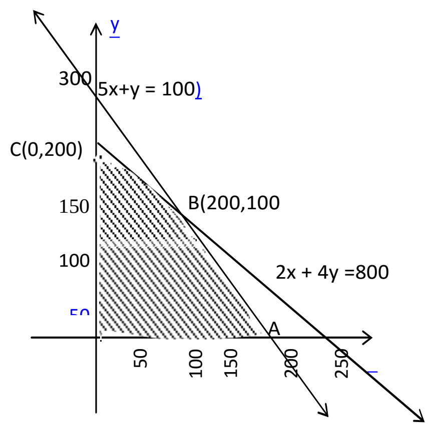
Clearly we have :	$x \geq 0$; $y \geq 0$	…(i)
Also	$2x + 4y \leq 800$	…(ii)
	$x + y \leq 300$	…(iii)

Let Z be sale function, $Z = 300x + 400y$

The problem reduces to maximize sale function subject to the conditions : $x \geq 0$, $y \geq 0$

$$2x + 4y \leq 800, \qquad x + y \leq 300 \qquad \ldots(A)$$

Let us draw the graph of system (A) and solution set of these inequations is the shaded region OABC and so the feasible region is the shaded region whose corner points are O(0, 0), A(300, 0), B(200, 100), C(0, 200). Since the maximum value of the sale function occurs only at the boundary point (s) and so we calculate the sale function at every point of the feasible region.



Boundary points of the feasible region	$S = 300x + 400y$
O(0, 0)	S = 300*0 + 400*0 = 0
A(300, 0)	S = 300*300 + 400*0 = Rs. 90, 000
B(200, 100)	S = 300*200 + 400*100 = Rs. 100000
C(0, 200)	S = 300*0 + 400*200 = Rs. 80, 000

Hence the maximum sale is Rs. 100000 at B(200, 100) and so company should produce 200 cord type and 100 cordless telephone sets.

1.4.4. Example. Solve the following LPP

Maximize $Z = 10x + 12y$

Subject to the constraints :

$$x + y \leq 5$$
$$4x + y \geq 4$$
$$x + 5y \geq 5$$
$$x \leq 4$$
$$y \leq 3.$$

Solution. It is a problem of mixed constraints. Constraints having greater than or equal to (≥) sign will have their feasible area to the right of their line while the constraints having less than a equal to (≤) sign will have their area to the left of their line.

The given linear constraints are :

$$x + y \leq 5 \qquad \ldots(1)$$

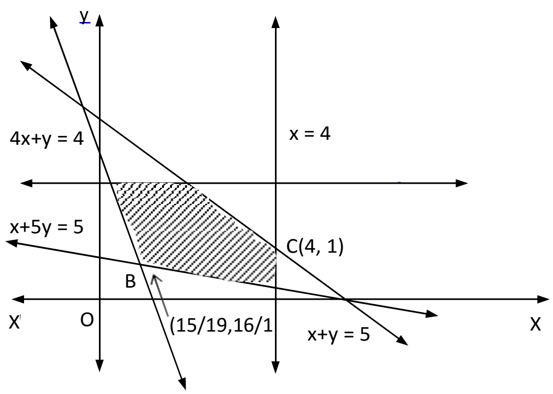
$$4x + y \geq 4 \qquad \ldots(2)$$

$$x + 5y \geq 5 \qquad \ldots(3)$$

$$x \leq 4 \qquad \ldots(4)$$

$$y \leq 3 \qquad \ldots(5)$$

Let us draw the graph of inequations (1), (2), (3), (4) and (5). The graph (or solution set) of these inequations is the shaded area (a polygen ABCDE) with vertices (1/4,3), (15/19, 16/19), (4,1/5), (4, 1), (2, 3) as shown in the figure below :



This shaded area is bounded by the five lines x + y = 5, 4x + y = 4, x + 5y = 5, x = 4 and y = 3.

Boundary points of the feasible region	Z = 10x + 12y
$A\left(\frac{1}{4}, 3\right)$	$Z = 10 \times \frac{1}{4} + 12 \times 3 = 38.5$
$B\left(\frac{15}{19}, \frac{16}{19}\right)$	$Z = 10 \times \frac{15}{19} + 12 \times \frac{16}{19} = 18$
$C\left(4, \frac{1}{5}\right)$	$Z = 10 \times 4 + 12 \times \frac{1}{5} = 42.4$
D(4,1)	$Z = 10 \times 4 + 12 \times 1 = 52$
E(2,3)	$Z = 10 \times 2 + 12 \times 3 = 56$

Since maximum value of Z is Rs. 56 at E, so optimum solution is x = 2, y = 3.

1.4.5. Example. If a young man rides his motor-cycle at 25 km per hour, he has to spend Rs. 2 per km on petrol, if he rides it at a faster speed of 40 km per hour, the petrol cost increases to Rs. 5 per km. He has Rs. 100 to spend on petrol and wishes to find what the maximum distance he can travel within one hour is? Express this as a linear programming problem and then solve it.

Solution. Let the young man ride x km at the speed of 25 km per hour and y km at the speed of 40 km per hour. Let f be the total distance covered, which is to be maximized.

Therefore, $f = x + y$ is the objective function.

Cost of travelling per km is Rs. 2 at the speed of 25 km per hour and cost of travelling per km is Rs. 5 at the speed of 40 km per hour.

Therefore, total cost of travelling = $2x + 5y$

Also Rs. 100 are available for petrol. Therefore,

$$2x + 5y \leq 100$$

Time taken to cover x km at the speed of 25 km per hour = $\frac{x}{25}$ hour

Time taken to cover y km at the speed of 40 km per hour = $\frac{y}{40}$ hour

Total time available = 1 hour

Therefore, we have

$$\frac{x}{25} + \frac{y}{40} \leq 1$$

or $$8x + 5y \leq 200$$

Also $x \geq 0$, $y \geq 0$

Therefore, we are to maximize

$$f = x + y$$

subject to the constraints

$$2x + 5y \leq 100$$

$$8x + 5y \leq 200$$

$$x \geq 0 , y \geq 0.$$

Consider a set of rectangular cartesian axes OXY in the plane.

It is clear that any point which satisfies $x \geq 0$, $y \geq 0$ lies in the first quadrant.

Let us draw the graph of the line $2x + 5y = 100$

For $x = 0$, $5y = 100$ or $y = 20$

For $y = 0$, $2x = 100$ or $x = 50$

Therefore, line meets OX in A(50, 0) and OY in L(0, 20)

Again we draw the graph of the line

$$8x + 5y = 200.$$

For $x = 0, 5y = 200$ or $y = 40$

For $y = 0, 8x = 200$ or $x = 25$

Therefore, line meets OX in B(25, 0) and OY in M(0, 40).

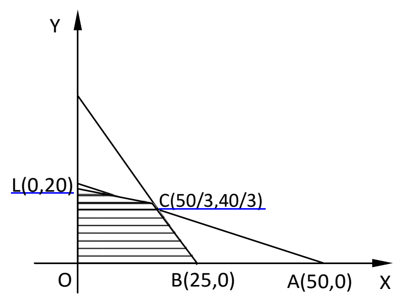
Since feasible region is the region which satisfies all the constraints,

Therefore, feasible region is the quadrilateral OBCL. The corner points are O(0, 0), B(25, 0), C(50/3,40,3), L(0, 20)

At O(0, 0) $f = 0 + 0 = 0$

At B(25, 0) $f = 25 + 0 = 25$

At C(50/3,40/3) $f = 50/3 + 40/3 = 30$



At L(0, 20) $f = 0 + 20 = 20$

Therefore, maximum value of f = 30 at (50/3,40/3).

Thus,the young man covers the maximum distance of 30 km when he rides 50/3 km at the speed of 25 km per hour and 40/3 km at the speed of 40 km per hour.

1.4.6. Example. A farmer decides to plant up to 10 hectares with cabbages and potatoes. He decided to grow at least 2, but not more than 8 hectares of cabbage and at least 1, but not more than 6 hectares of potatoes. If he can make a profit of Rs. 1500 per hectare on cabbages and Rs. 2000 per hectare on potatoes how should he plan his farming so as to get the maximum profit ? (Assuming that all the yield that he gets is sold.)

Solution. Suppose the farmer plants x hectares with cabbages and y hectares with potatoes.

Then the constraints are

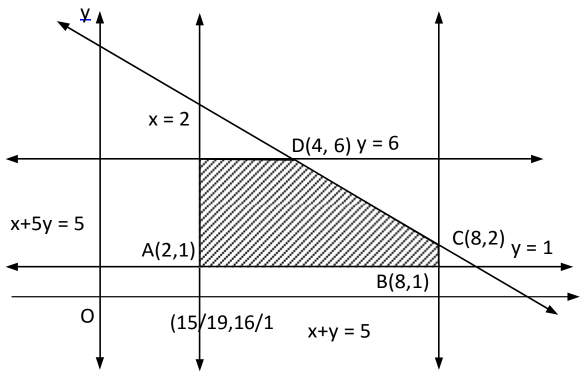
$2 \leq x \leq 8$ …(1)

$1 \leq y \leq 6$ …(2)

$x + y \leq 10$ …(3)

and $P = 1500\,x + 2000y$ …(4)

We draw the lines $x = 2$, $x = 8$, $y = 1$, $y = 6$ and $x + y = 10$.



The vertices of the solution set ABCDE are

A(2, 1), B(8, 1), C(8, 2), D(6, 4) and E(2, 6)

Now

at A(2, 1), P = 1500 (2) + 2000(1) = 3000 + 2000 = 5000

at B(8, 1), P = 1500 (8) + 2000(1) = 12000 + 2000 = 14000

at C(8, 2), P = 1500 (8) + 2000(2) = 12000 + 4000 = 16000

at D(4, 6), P = 1500 (4) + 2000(6) = 6000 + 12000 = 18000

at E(2, 6), P = 1500 (2) + 2000(6) = 3000 + 12000 = 15000.

Hence in order to maximise profit the farmer plants 4 hectares with cabbages and 6 hectares with potatoes.

1.4.7. Example. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least four times as many passengers prefer to travel by economy class than by the first class. Determine how many each type of tickets must be sold in order to maximise the profit for the airline. What is the maximum profit ?

Solution. Let the number of first class tickets and Economy class tickets sold by the Airline be x and y respectively.

Maximum capacity of passengers is 200 i.e. $x + y \leq 200$ …(i)

At least 20 seats of first class are reserved $x \geq 20$ …(ii)

At least 4 x seats of Economy class are reserved $y \geq 4x$ …(iii)

Let P the profit function, $P = 400x + 300y$

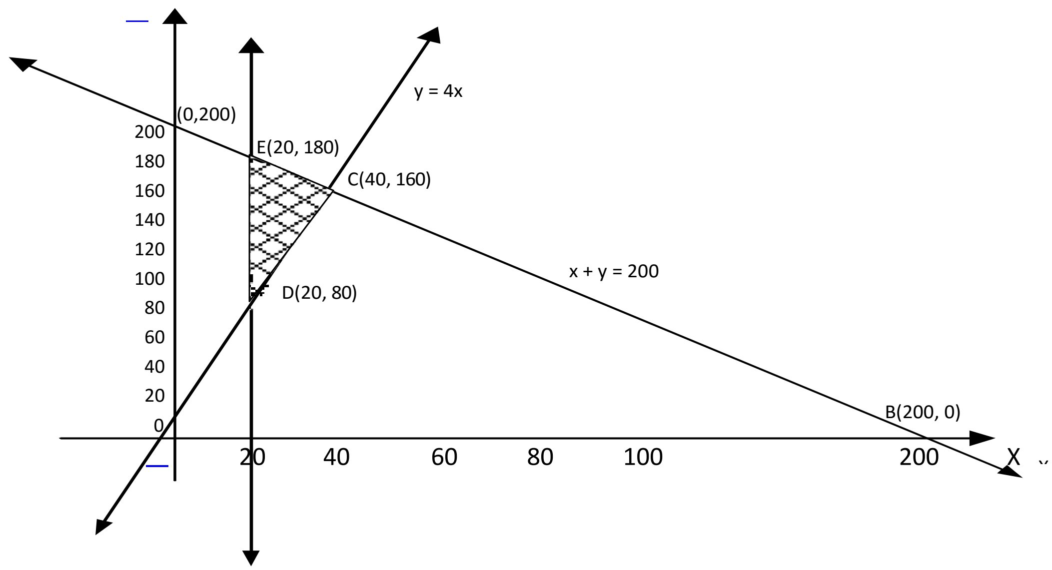
Therefore, the problem reduces to maximize P subject to the constraints $x \geq 20$; $y \geq 4x$ and $x + y \leq 200$.

Let us find out the solution set of the inequations $x \geq 20$; $y \geq 4x$ and $x + y \leq 200$.

The triangular shaded region CDE is the feasible region and its vertices are :

C(40, 160), D(20, 80), E(20, 180)

Since the maximum or minimum value occurs at the boundary point (s) and we calculate the profit function P at every vertex of the feasible region.



Boundary points of the feasible region	P = 400x + 300y
C(40, 160)	P = 400*40 + 300*160 = Rs. 64,000
D(20, 80)	P = 400*20 + 300*80 = Rs. 32,000
E(20, 180)	P = 400*20 + 300*180 = Rs. 62,000

Therefore, the maximum profit Rs. 64000 is obtained at C(40, 160) and so Airline should sell 40 tickets of first and 160 tickets of economy class.

1.4.8. Minimisation Case.

In such questions, value of the objective function is to be minimised. Generally, in the questions involving cost, distance, expenses risk etc. our objective to keep their value least.

Generally in case of maximisation, we use the constraints of less than or equal to (≤) type and in case of minimisation, we use constraints of greater than a equal to (≥) type. But same some times, we also use mixed constraints (both ≥ and ≤).

1.4.9. Example. Minimize

$$C = x + y$$

subject to

$$3x + 2y \geq 12$$

$$x + 3y \geq 11$$

$$x \geq 0$$

$$y \geq 0$$

Solution. Let us solve graphically the following inequations

$3x + 2y \geq 12$

$x + 3y \geq 11$

$x \geq 0$

$y \geq 0$

Changing the inequalities into equations, we get

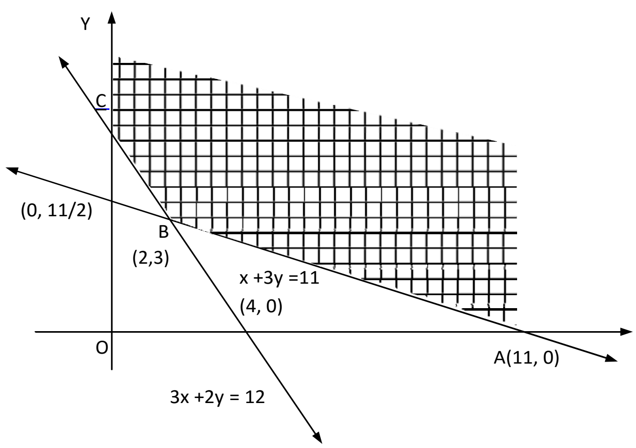
$3x + 2y = 12$	$x + 3y = 11$
For x=0, y=6	For x=11, y=0
For x=4, y=0	For x=0, y=11/3

Shaded region ABC is the required feasible region of the above stated inequations.

Boundary points of the feasible region are :

A(11, 0), B(2, 3), C(0, 6).

Since minimum or maximum always occurs only at boundary point (s) and so will calculate the cost (C) at every boundary point of the feasible region.



Boundary point of the feasible region	C = x + y
A(11, 0)	11 +0 = Rs. 11
B(2, 3)	2 + 3 = Rs. 5 (Mini. Cost)
C(0, 6)	0 + 6 = Rs. 6

Hence minimum cost is at the point B(2, 3) and minimum cost is Rs. 5.

1.4.10. Example. Find the maximum and minimum values of the function 2 = 3x + y

Subject to the constraint

$x + y \leq 2$

$4x + y \leq 5$

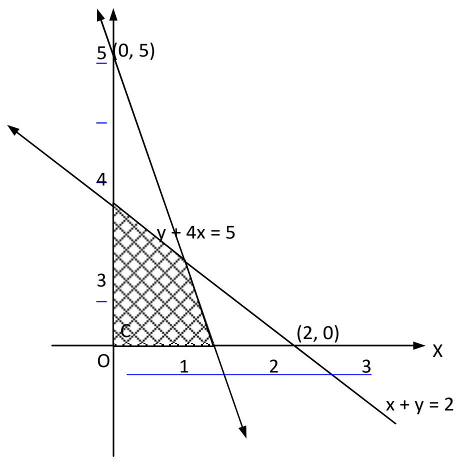
$x, y \geq 0$

Solution. Let us first change the inequalities into equation

x + y = 2	4x + y = 5
For x=0, y=2	For x=0, y=5
For x=2, y=0	For x=5/4, y=0

Solution set of the above inequations is the required shaded feasible region OABC whose boundary points are O(0, 0), A(5/4,0), B(1, 1), C(0, 2).

Since the minimum or maximum value of C occurs only at the boundary point (s) and so let us calculate the value of Z at every vertex of the feasible region OABC.



Boundary points of the feasible region	Z = 3x + y
O(0, 0)	Z = 3* 0 + 0 = Re. 0 (Minimum cost)
A(5/4 , 0)	Z = 3*(5/4) + 0 = Rs. 3.75
B(1, 1)	Z = 3*1 + 1 = Rs. 4
C(0, 2)	Z = 3*0 + 2 = Rs. 2

So the maximum value is Rs. 4 and the minimum value Rs. 0 for maximisation x = 1, y = 1 and for minimization x = 0, y = 0.

1.4.11. Example. Minimise P = 2x + 3y, subject to the conditions

$$x \geq 0,\ y \geq 0,\ 1 \leq x + 2y \leq 10.$$

Solution. We have

$x \geq 0$	…(1)	$y \geq 0$	…(2)
$x + 2y \geq 1$	…(3)	$x + 2y \leq 10$	…(4)

and P = x + 3y

We find out the solution set (convex polygon), where (1) - (4) are true.

For this, we draw the graph of the lines

$x = 0, y = 0, x + 2y = 1, x + 2y = 10.$

The shaded portion is the feasible region of the constraints.

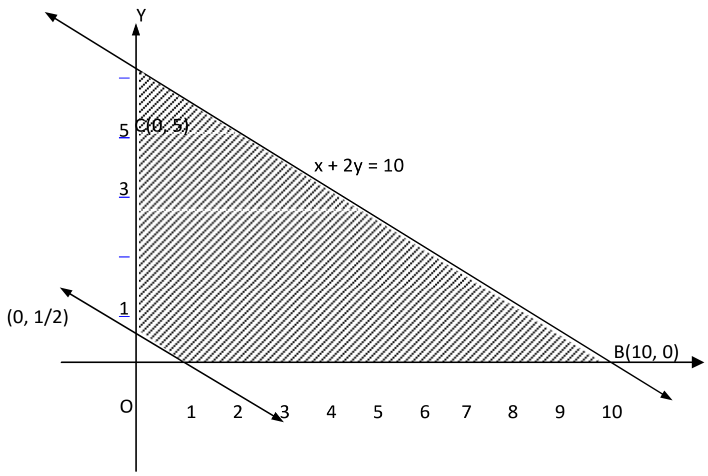
Now

at A(1, 0), $P = 2(1) + 3(0) = 2$

at B(10, 0), $P = 2(10) + 3(0) = 20$

at C(0, 5), $P = 2(0) + 3(5) = 15$

at D(0,1/2), $P = 2(0) + 3(1/2) = 3/2$



Since the minimum value is at D, so the optimal solution is $x = 0$, $y = 1/2$.

1.4.12. Example. Find the maximum and minimum value of

$$Z = x + 2y$$

subject to

$$2x + 3y \leq 6$$

$$x + 4y \leq 4$$

$$x, y \geq 0$$

Solution. We are maximize and minimize

$$Z = x + 2y$$

Subject of the constraints $2x + 3y \leq 6$

$$x + 4y \leq 4$$

$$x, y \geq 0$$

First, we draw the graph of the line $2x + 3y = 6$.

For $x = 0$, $3y = 6$, or $y = 2$

For $y = 0$, $2x = 6$, or $x = 3$

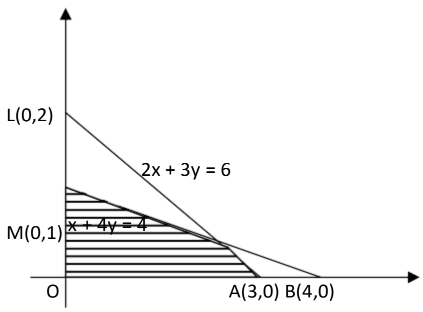
Therefore, line meets OX in A(3, 0) and OY in L(0,2).

Draw the graph of line x + 4y = 4

For x = 0, 4y = 4 or y = 1

For y = 0, x = 4

Therefore, line meets OX in B(4, 0) and OY in M(0,1).



Since feasible region is the region which satisfies all the constraints. OACM is the feasible region. The corner points are O(0, 0), A(3, 0), C(12/5, 2/5), M(0, 1).

At O(0, 0), f = 0 + 0 = 0

At A(3, 0), f = 3 + 0 = 3

At C(12/5, 2/5), f = 12/5 + 4/5 = 16/5 = 3.2

At D(0,1), f = 0+2 =2

Therefore, minimum value = 0 at (0,0) and maximum value = 3.2 at (12/5, 2/5).

1.5. Check Your Progress.

Draw the diagrams of the solution sets of the following (1 - 3) linear constraints :

1. $3x + 4y \geq 12$, $4x + 7y \leq 28$, $x \geq 0$, $y \geq 1$.
2. $x + y \leq 5$, $4x + y \geq 4$, $x + 5y \geq 5$, $x \leq 4$, $y \leq 3$.
3. $x + y \geq 1$, $y \leq 5$, $x \leq 6$, $7x + 9y \leq 63$, $x, y \geq 0$.
4. Draw the graph of the equation $2x + 3y \leq 35$.

5. Verify that the solution set of the following constraints is empty : $3x + 4y \geq 12$, $x + 2y \leq 3$, $x \geq 0$, $y \geq 1$.

6. Verify that the solution set of the following constraints : $x - 2y \geq 0$, $2x - y \leq -2$ is not empty and is unbounded.

7. Draw the diagram of the solution set of the linear constraints

(i)	(ii)
$3x + 2y \leq 18$	$2x + y \geq 4$
$x + 2y \leq 10$	$3x + 5y \geq 15$
$x \geq 3$, $y \geq 0$	$x \geq 0$, $y \geq 0$

8. Exhibit graphically the solution set of the linear constraints

$x + y \geq 1$

$y \leq 5$

$x \leq 6$

$7x + 9y \leq 63$

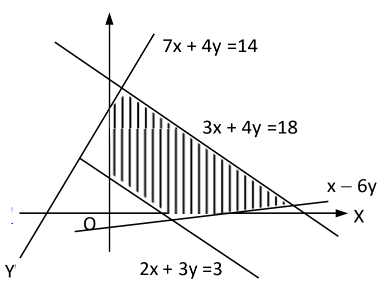
$x, y \geq 0.$

9. Verify that the solution set of the following linear constraints is empty :

(i) $x - 2y \geq 0$	(ii) $3x + 4y \geq 12$
$2x - y \leq -2$	$x + 2y \leq 3$
$x \geq 0, y \geq 0$	$y \geq 1, x \geq 0$

10. Verify that the solution set of the following linear constraints is unbounded :

$3x + 4y \geq 12, \; y \geq 1, \; x \geq 0$



11. Find the linear constraints for which the shaded area in the figures below is the solution set :

12. Find the maximum and minimum value of $2x + y$ subject to the constraints

$2x + 3y \leq 30, \; x - 2y \leq 8, \; x \geq 0, \; y \geq 0.$

13. Solve by graphical method :

(i) Minimize $Z = 3x_1 + 2x_2$ subject to the constrains

$-2x_1 + x_2 \leq 1, \; x_1 \leq 2, \; x_1 + x_2 \leq 3, \; x_1, x_2 \geq 0.$

(ii) Find the maximum value of $z = 5x_1 + 3x_2$ subject to the constraints

$3x_1 + 5x_2 \leq 15, \; 5x_1 + 2x_2 \leq 10, \; x_1, x_2 \geq 0.$

(iii) Find the maximum value of $z = 2x_1 + 3x_2$ subject to the constraints

$x_1 + x_2 \leq 1, \; 3x_1 + x_2 \leq 4, \; x_1, x_2 \geq 0.$

(iv) Find the minimum value of $3x + 5y$ subject to the constraints

$-2x + y \leq 4, \; x + y \geq 3, \; x - 2y \leq 2, \; x, y \geq 0.$

14. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space of at most 20 items. A fan costs him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming he can sell all the items that he can buy, how should he invest his money in order to maximise his profit ? Also find the maximum profit.

15. A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units per kg. of vitamin A and I unit per kg. of vitamin C, while the food II contains 1 unit per kg. of Vitamin A and 2 units per kg. of vitamin C. It costs Rs. 5 per kg. to purchase food I and Rs. 7 per kg. to purchase food II Find the minimum cost of such mixture and the quantity of the each of the foods.

15. A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine A and 3 on machine B to produce a package of nuts while it takes 32 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs. 2.50 per package on nuts and Rs. 1.00 per package on bolts. How many packages of each should he produce each day so as maximise his profit, if he operates his machines for at the most 12 hours a day ?

16. A sports factory prepares cricket bats and hockey sticks. A cricket nat bat takes 2 hours of machine time and 3 hours of craftsman's time. A hockey stick take 3 hours of machine time and 2 hours of craftsman's time. The factory has 90 hours of machine time and 85 hours of craftsman's time. What number of bats and sticks must be made if the factory is to work at full capacity ? If the profit on a bat is Rs. 3 and on a stick it is Rs. 4, find the maximum profit.

17. A trader deals in sewing machines and transistors. It has capacity to store at the most 30 pieces and he can invest Rs. 4500. A machine costs him Rs. 250 each and transistor costs him Rs. 100 each. The profit on machine Rs. 40 and on a transistor it is Rs. 25. Find number of sewing machines and transistors to take max. profit.

18. Every gram of wheat provides 0.1 g of proteins and 0.25 g of carbohydrates. The corresponding values for rice are 0.05 g and 0.5 g respectively. Wheat costs Rs. 2 per kg and rice Rs. 8. The minimum daily requirements of protein and carbohydrates for an average child are 50 g and 200 g respectively. In what quantities should wheat and rice be mixed in the daily diet to provide the minimum daily requirements of protein and carbohydrates at minimum cost.

19. A toy company manufactures two types of dolls, a basic version-doll A and a deluxe version-doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2,000 per day if it produced only the basic version. The supply of plastic is sufficient t o produce 1,500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs. 3.00 and Rs. 5.00 per doll respectively on doll A and B ; how many of each should be produced pe r day in order to maximize profit ?

20. Smita goes to the market to purchase battons. She needs at least 20 large battons and at least 30 small battons. tHe The shopkeeper sells battons in two forms (i) boxes and (ii) cards. A box contains then large and five small battons and a card contains two large and five small battons. Find the most economical way in which she should purchase the battons, if a box costs 25 paise and a card 10 paise only.

21. Vikram has two machines with which he can manufacture either bottles or tumblers. The first of the two machines has to be used for one minute and the second for two minutes in order to manufacture a bottle and the two machines have to be used for one minute each to manufacture a tumbler. During an hour the two machines can be operated for at the most 50 and 54 minutes respectively. Assuming that he can sell as many bottles and tumblers as the can produce, find how many of bottles and tumblers he should manufacture so that his profit per hour is maximum being given that the gets a profit of ten paise per bottle and six paise per tumbler.

22. A company produces two types of presentation goods A and B that require gold and silver. Each unit of type A requires 3 gms. of silver and 1 gm of gold while that of B requires 1 gm. of silver and 2 gm. of gold. The company can produce 9 gms. of silver and 8 gms. of gold. If each unit of type A brings a profit of Rs. 40 and that of type B Rs. 50 determine the number of units of each type that the company should produce to maximize the profit. What is the maximum profit ?

23. A gardener uses two types of fertilizers I and II. Type I consists of 10% nitrogen and 6% phosphoric acid while type II consists of 5% nitrogen and 10% phosphoric acid. He requires at least 14 kg. of both nitrogen and phosphoric acid for his crop. If the type I fertilizer costs Rs. 0.60 per kg and type II costs Rs. 0.40 per kg., how many kilograms of each fertilizer he should use so as to minimise the total cost. Also find the minimum cost.

1.6. Summary. In this chapter, we discussed about solving the given maximization and minimization problems using graphical solutions and observed that there are variety of regions in which solution exists and sometimes does not exists.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.

2

Simplex Method and Transportation Problem

Structure

2.1. Introduction. Main limitation of graphical method in solving linear programming problem is that using this method, we can solve problems involving two variables only. In real life, we may have to solve the problems involving more than two variables. In such situation, we can't use this method. Simplex method is a technique with the help of which we can find solutions to such problems.

2.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Simplex Method.

(ii) Duality.

(iii) Transportation Problem.

2.1.2. Keywords. Constraints, Unique Solution, Transportation.

2.2. Conditions for application of Simplex Method.

To apply this techniques the following two conditions must be satisfied

1. R.H.S of every constraint is equality must be non-negative. If it is negative in any in equality, it is made positive by multiplying both sides of inequality by (-1). For example if we are given the constraint $2x_1 - 5x_2 \geq -10$. Then we can rewrite it as $-2x_1 + 5x_2 \leq 10$) Note that when we multiply both sides by (-1), the sign of inequality changes.
2. Decision variables like x_1, x_2 would also be non-negative. If it is given that any decision variable is unrestricted in sign, it is expressed as difference of two non-negative variables, For example if it is give that x_3 is unrestricted in sight, we can write it as $x_3 = x_4 - x_5$.

Steps involued in simplex method

First write the objective function. It is either maximisation or minimisation.

For example, $Max\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ or $Min\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Write the constraint inequalities with proper signs.

For example, $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$ or $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$

and so on.

These in equalities are converted into equation by introducing slack and artificial variable. Slack variables now the left over quantity of the resources. Artificial variables have no real value. They are introduced first to solve the problem.

Important Note. If the in equations are of less than or equal to ($\leq$) sign only slack variables are introduced. But if the in equations are of greater than or equal ($\geq$ sign), both slack and artificial variables are introduced.

2.2.1. Example. Two given constraints are

$$2x_1 + 3x_2 \leq 60 \text{ and } 4x_1 + x_2 \geq 40$$

We will rewrite them as

$$2x_1 + 3x_2 + S_1 = 60 \text{ and } 4x_1 + x_2 - S_2 + A_1 = 40$$

In less then constraints, slack variables (S) will have positive sign and in more than constraints, they will have negative sign. These equations are then presented in the form of a matrix where format is shown below with the help of an example.

2.2.2. Example Maximise $Z = 22\ x_1 + 18\ x_2$

Subject to $x_1 + x_2 \leq 20$

$360\ x_1 + 240\ x_2 \leq 5760$

change the in equations into equations $x_1 + x_2 + S_1 = 20$

$360\ x_1 + 240x_2 + S_2 = 5760$

Table 1.

Basic	C_i	x_1	x_2	S_1	S_2	B	
S_1	0	1	1	1	0	20	Constraint values
S_2	0	360	240	0	1	5760	
C_j		22	18	0	0	Co-efficient values from constraint equation	
Zj		0	0	0	0		
Cj - Zj		22	18	0	0		

Cj is the contribution/unit of each variable shown in the objective function. Slack variables have zero contribution. Zj shows the total contribution of various variables at any given stage.

The row showing (Cj - Zj) is called the index row. This row shows how much profit is foregone by not producing one unit of a product etc.

Remember. An optimal solution is searched when all the values of index row become zero or negative. Now for finding the optimal solution, we consider two cases -(i) Maximisation case (ii) Minimisation case.

2.2.3. Maximisation Case. Let us reconsider the above example.

1) (Finding the initial feasible solution) our problem is

Max. $Z = 22\,x_1 + 18\,x_2$

Subject to $x_1 + x_2 \leq 20$

$360\,x_1 + 240\,x_2 \leq 5760$

or $x_1 + x_2 + S_1 = 20$ ...(1)

$360\,x_1 + 240\,x_2 + S_2 = 5760$ ...(2)

Initially, put $x_1 = 0$, $x_2 = 0$. So from (1) $S_1 = 20$ and from (2) $S_2 = 5760$.

In the initial solution, we assume that we are not producing any quantity of either of the products. So the resources remain fully unutilized. That whey in equation (1) we get $S_1 = 20$ and in (2) we get $S_2 = 5760$.

This solution is shown in the above table (in the term of matrix.)

Find the highest positive value in the row (Zj - Cj). The variable of the column to which this value corresponds will enter the solution. Divide the constraint values (b's) by the element of this column to find the ratio (bij/aij). Choose the ratio which has minimum positive value and find the row of this ratio. The basic variable of this row will leave the solution and these above variable will replace this variable consider our example

Basic	C_i	x_1	x_2	S_1	S_2	b	b/a
S_1	0	1	1	1	0	20	20/1 = 20
S_2	0	360	240	0	1	5760	5760/360 = 16 →Outgoing variable
C_j		22	18	0	0		Co-efficient values from constraint equation
Zj		0	0	0	0		
Cj - Zj		22	18	0	0		

Incoming variable

So now x_1 will replace S_2

The element which belongs to both key column and key row is called key element. Now divide all elements of key row by key column like

Key row = 360 240 0 1 5760

Divide all the values by 360, we get

0 2/3 0 1/360 16

After using matrix operations, all other elements of key row are made equal to zero like

1st row	1	1	1	0	20
2nd row	1	2/3	0	1/360	16

R_1 ® R_1 - R_2 to get

1st row	0	1/3	1	-1/360	4
2nd row	1	2/3	0	1/360	16

After all these changes, the new matrices will be as follow

Table 2

		x_1	x_2	S_1	S_2	B	Ratio
0	S_1	0	1/3	1	-1/360	4	12 Key row
22	x_1	1	2/3	0	1/360	16	24
C_i		22	18	0	0	0*4+22*16=352 Total profit at this stage	
Zj		0	44/3	0	22/360		
Cj – Zj		0	10/3	0	-22/360		

Because still are positive value remains in the (C_j - Z_j) row, so we have get to obtain optimal solution. Now we will repeat steps 2 and 3 and repeat them till all the values in the (C_j - Z_j) row become be zero or negative.

The new table will be as follows:

Table 3

	x_1	x_2	S_1	S_2	b
18 x_2	0	1	3	-1/120	12
22 x_1	1	0	-2	1/120	8
C_i	22	18	0	0	18*12+22*8=392
Zj	22	18	10	4/120	Total profit at this stage
Cj – Zj	0	0	-10	-4/120	

Now there is no positive value in the index row, so we have obtained optimal solution. The optimal solution is $x_1 = 8$, $x_2 = 12$ and maximum profit Z = Rs. 392 (obtained from the resources column)

2.2.4. Example. A firm produces three products A, B and C, each of which passes through three departments : Fabrication, Finishing and Packaging. Each unit of product A requires 3, 4 and 2; a unit of product B requires 5, 4 and 4, while each unit of product C requires 2, 4 and 5 hours respectively in the three departments. Every day, 60 hours are available in the fabrication department, 72 hours in the finishing department and 100 hours in the packaging department. The unit contribution of product A is Rs 5, of product B is Rs. 10, and of product C is Rs. 8.

Required :

a) Formulate the problem as an LPP and determine the number of units of each of the products, that should be made each day to maximise the total contribution. Also determine if any capacity would remain unutilized.

Solution. Let x_1, x_2 and x_3 represent the number of units of products A, B and C respectively. The given problem can be expressed as a LPP as follows :

Maximise	$Z = 5x_1 + 10x_2 + 8x_3$	Contribution
Subject to	$3x_1 + 5x_2 + 2x_3 \leq 60$	Fabrication hours
	$4x_1 + 4x_2 + 4x_3 \leq 72$	Finishing hours
	$2x_1 + 4x_2 + 5x_3 \leq 100$	Packaging hours
	$x_1,x_2,x_3 \geq 0$	

Introducing slack variables, the augmented problem can be written as

Maximise $Z = 5x_1 + 10x_2 + 8x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$3x_1 + 5x_2 + 2x_3 + S_1 = 60$$

$$4x_1 + 4x_2 + 4x_3 + S_2 = 72$$

$$2x_1 + 4x_2 + 5x_3 + S_3 = 100$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

The solution to the problem using simplex algorithm is contained in Tables 1 to 3.

Simplex Table 1: Initial Solution

Basic		x_1	x_2	x_3	S_1	S_2	S_3	b	Ratio b_i/a_{ij}	
S_1	0	3	2	5*	1	0	0	60	12	Outgoing variable (key row)
S_2	0	4	4	4	0	1	0	72	18	
S_3	0	2	4	5	0	0	1	100	15	
C_i		5	10	8	0	0	0			
Zj		0	0	0	0	0	0			
Cj - Zj		5	10	8	0	0	0			

5* is the key element

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_2	x_3	S_1	S_2	S_3	b	Ratio b_i/a_{ij}	
x_2	10	3/5	1	2/5	1/5	0	0	12	30	Outgoing variable (key row)
S_2	0	8/5	0	12/5	-4/5	1	0	24	10	
S_3	0	-2/5	0	17/5	-4/5	0	1	52	260/17	
C_i		5	10	8	0	0	0			
Zj		6	10	4	2	0	0			
Cj - Zj		-1	0	4	-2	0	0			

Simplex Table 3: Optimal Solution

Basic		x_1	x_2	x_3	S_1	S_2	S_3	b_i
x_2	10	1/3	1	0	1/3	-1/6	0	8
x_3	8	2/3	0	1	-1/3	5/12	0	10
S_3	0	-8/3	0	0	1/3	-17/12	1	18
C_i		5	10	8	0	0	0	
Solution		0	8	10	2	0	18	
Cj - Zj		-11/3	0	0	-2/3	-5/3	0	

According to the Simplex Table 3, the optimal solution is : $x_1 = 0$, $x_2 = 8$, $x_3 = 10$. Thus, it calls for producing 8 and 10 units of products B and C respectively, each day. This mix would yield a contribution of 5 * 0 + 10* 8 + 8 *10 = Rs. 160. S_3 being equal to 18, an equal number of hours shall remain unutilized in the packaging department.

2.2.5. Example. Solve the following L.P.P.

Maximise $Z = 40000\,x_1 + 55000x_2$

Subject to $1000\,x_1 + 1500x_2 \leq 20000$

$x_1 \leq 12$

$x_2 \geq 5$

$x_1, x_2 \geq 0$

Solution. By changing the inequations into equations by adding surplus and artificial variables, the form of the problem is changed as :

Maximise $Z = 40000x_1 + 15000x_2 + 0.S_1 + 0S_2 + 0S_3 - M.A.$

Subject to $1000x_1 + 1500x_2 + S_1 = 2000$

$x_1 + S_2 = 12$

$x_2 - S_3 + A_1 = 5$

$x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$

The solution to this problem is shown in tables 1 to 3

Simplex Table 1: Initial Solution

Basic		x_1	x_2	S_1	S_2	S_3	A_1	b_i	b_i/a_{ij}
S_1	0	1000	1500	1	0	0	0	20000	40/3
S_2	0	1	0	0	1	0	0	12	-
A_1	-M	0	1*	0	0	-1	1	5	5 (key row)
C_j		40000	55000	0	0	0	-M		
Z_j		0	-M	0	0	M	-M	-5M	
$C_j - Z_j$		40000	55000+M	0	0	-M	0		

(Incoming variable)
Key column

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_2	S_1	S_2	S_3	A_1	b	b_i/a_{ij}
S_1	0	1000	0	1	0	1500*	-1500	12500	1250/1500 (key row)
S_2	0	1	0	0	1	0	0	12	-
x_2	55000	0	1	0	0	-1	1	5	-
C_j		40000	55000	0	0	0	-M		
Z_j		0	55000	0	0	-55000	55000	275000	
$C_j - Z_j$		40000	0	0	0	55000	-M-55000		

(key column)

*key element

Simplex Table 3: Non-optimal Solution

Basic		x_1	x_2	S_1	S_2	S_3	A_1	B	b_i/a_{ij}
S_3	0	2/3	0	1/1500	0	1	-1	25/3	25/212 (key row)
S_2	0	1	0	0	1	0	0	12	-
x_2	55000	2/3	1	1/1500	0	0	0	40/3	20
C_j		40000	55000	0	0	0	-M		
Z_j		11000/3	55000	110/3	0	0	0	2200000/3	
$C_j - Z_j$		10000/3	0	-110/3	0	0	-M		
		(key column)							

Simplex Table 4: Optimal Solution

Basic		x_1	x_2	S_1	S_2	S_3	A_1	B
S_3	0	0	0	1/1500	-2/3	1	-1	1/3
x_1	40000	1	0	0	1	0	0	12
x_2	55000	0	1	1/1500	-2/3	0	0	16/3
C_j		40000	55000	0	0	0	-M	
Z_j		40000	55000	110/3	10000/3	0	0	2320000/3
$C_j - Z_j$		0	0	-110/3	-10000/3	0	-M	

So optimal solution is $x_1 = 12$, $x_2 = 16/3$ and Z = 2320000/3.

2.2.6. Minimization Case.

Steps involved in finding the minimum value of objective functions are same as in case of maximization. Same fundamental differences should be taken case of which are as follows :

1. In the table showing initial solution, we will take highest negative value not the highest positive value. The column which has this value is the key column.
2. In problems of minimisation, if we use artificial variables then they will have a weight of +M whereas in problems of maximization, they have negative weight -M.
3. While going for optimal solution, these artificial variables leave the solution. If they are in the solution in the final table, it means that the given problem has no feasible solution.
4. When all the values in the index row are zero a positive, optimal solution is reached.

2.2.7. Example. To improve the productivity of land, a framer is advised to use at least 4800 kg. of phosphate fertilizer and not less than 7200 kg. of nitrogen fertilizer. There are two sources to object these fertilizers mixture A and B. Both of these are available in bags of 100 kg. each and their cost per bag are Rs. 40 and Rs. 24 respectively. Mixture A contains 20 kg. phosphate and 80 kg. nitrogen while their respective quantities in mixture B are 80 kg. and 50 kg. Formulate this as an LPP and determine how many bags of each type of mixture the farmer should buy in order to obtain the required fertilizer at minimum cost.

Solution. Let x1 be number of bags of mixture A and x2 be the number of bags of mixture B. So now the problem can be written as

Minimise $Z = 40x_1 + 24x_2$ Total cost

Subject to

$20x_1 + 50x_2 \geq 4800$ Phosphate Requirement

$80x_1 + 50x_2 \geq 7200$ Nitrogen Requirement

$x_1, x_2 \geq 0$

After introducing the slack + artificial variables, the above problem can be rewritten as :

Minimise $Z = 40x_1 + 24x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to

$20x_1 + 50x_2 - S_1 + A_1 = 4800$

$80x_1 + 50x_2 - S_2 + A_2 = 7200$

Simplex Table 1: Initial Solution

Basic		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
A_1	M	20	50*	-1	0	1	0	4800	96 (key row)
A_2	M	80	50	0	-1	0	1	7200	144
C_j		40	24	0	0	M	M		
Z_j		100M	100M	-M	-M	M	M	12000M	
$C_j - Z_j$		40-100M	24-100M	M	M	0	0		

(Key column)

*key element

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
x_2	24	2/5	1	-1/50	0	1/50	0	96	240
A_2	M	60	0	1	-1	-1	1	2400	40 (key row)
C_j		40	24	0	0	M	M		
Z_j		48/5+60M	24	M-24/50	-M	-M+24/50	M	2304+	
$C_j - Z_j$		152/2-60M	0	12/25-M	M	2M-12/25	0	2400M	

(Key column)

Simplex Table 3: Non-optimal Solution

Basic		x_1	x_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
x_2	24	0	1	-2/75	1/150	2/75	-1/150	80	-3000
x_1	40	1	0	1/60*	-1/60	-1/60	1/60	40	2400(key row)
C_j		40	24	0	0	M	M		
Z_j		40	24	2/75	-38/75	-2/75	-38/75	3520	
$C_j - Z_j$		0	0	-2/75	38/75	M+2/75	M+38/75		
				Key column					

Simplex Table 4: Optimal Solution

Basic		x_1	x_2	S_1	S_2	A_1	A_2	b_i
x_2	24	8/5	1	0	-1/150	0	1/50	144
S_1	0	60	0	1	-1	-1	1	2400
C_j		40	24	0	0	M	M	
Z_j		192/5	24	0	-12/25	M	12/25	3456
$C_j - Z_j$		8/5	0	0	12/25	0	M-12/25	

Since all the values of the index row are zero or positive, so we have got optimal solution. The optimal solution $x_2 = 144$, $x_1 = 0$ and Z = Rs. 3456.

2.2.8. Example. A finished product must weigh exactly 150 grams. The two raw materials used in manufacturing the product are A, with a cost of Rs. 2 per unit and B with a cost of Rs. 8 per unit. At least 14 units of B not more than 20 units of A must be used. Each unit of A and B weighs 5 and 10 grams respectively. How much of each type of raw material should be used for each unit of the final product in order to minimise the cost ? Use Simplex method.

Solution. The given problem can be expressed as LPP as

Minimise $Z = 2x_1 + 8x_3$

Subject to $5x_1 + 10x_3 = 150$

$x_1 \leq 20$

$x_2 \geq 14$

$x_1, x_2 \geq 0$

Substituting $x_2 = 14 + x_3$ and introducing necessary slack and artificial variables, we have, Minimise $Z = 2x_1 + 8x_3 + 112 + MA_1 + 0x_4$

Subject to $5x_1 + 10x_3 + A_1 = 10$

$x_1 + x_4 = 20$

$x_1, x_3, x_4, A_1 \geq 0$

The solution is contained in the following tables.

Simplex Table 1: Initial Solution

Basic		x_1	x_3	A_1	x_4	b_i	b_i/a_{ij}
A_1	M	5	10*	1	0	10	1(key row)
x_4	0	1	0	0	1	20	-
C_j		2	8	M	0		
Z_j		5M	10M	M	0	10M	
$C_j - Z_j$		2-5M	8-10M	0	0		

(Key column)

*key element

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_3	A_1	x_4	b_i	b_i/a_{ij}
x_3	8	½*	1	1/10	0	1	2(key row)
x_4	0	1	0	0	1	20	20
C_j		2	8	M	0		
Z_j		4	8	8/10	0	8	
$C_j - Z_j$		-2	0	M-8/10	0		

(Key column)

*key element

Simplex Table 3: Optimal Solution

Basic		x_1	x_3	A_1	x_4	b_i
x_1	2	1	2	1/5	0	2
x_4	0	0	-2	-1/5	1	18
C_j		2	8	M	0	
Z_j		0	4	2/5	0	4
$C_j - Z_j$		0	4	M-2/5	0	

Thus, the optimal solution is : x_1 = 2 units, x_3 = 14 + 0 = 14 units, total cost = 2 *2 + 8* 14 = Rs. 116.

2.2.9. Example. A company produces three products, P_1, P_2 and P_3 from two raw materials A and B, and labour L. One, unit of product P_1 requires one unit of A, 3 units of B and 2 units of L. One unit of product P_2 requires 2 units of A and B each, and 3 units of L, while one units of P_3 needs 2 units of A, 6 units of B and 4 units of L. The company has a daily availability of 8 units of A, 12 units of B and 12 units of L. It is further known that the unit contribution margin for the products is Rs. 3, 2 and 5 respectively for P_1, P_2 and P_3. Formulate this problem as a linear programming problem, and then solve it to determine the optimum product mix. Is the solution obtained by you unique ? Identify an alternate optimum solution, if any.

Solution. If x_1, x_2 and x_3 be the output of the products P_1, P_2 and P_3, respectively, we may express the linear programming formulation as follows :

Maximise $Z = 3x_1 + 2x_2 + 5x_3$ Contribution

Subject to $x_1 + 2x_2 + 2x_3 \leq 8$ Material A

$3x_1 + 2x_2 + 6x_3 \leq 12$ Material B

$2x_1 + 3x_2 + 4x_3 \leq 12$ Labour

$x_1, x_2, x_3 \geq 0$

Introducing slack variables S_1, S_2 and S_3, we may write the problem as follows :

Maximise $Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to $x_1 + 2x_2 + 2x_3 + S_1 = 8$

$3x_1 + 2x_2 + 6x_3 + S_2 = 12$

$2x_1 + 3x_2 + 4x_3 + S_3 = 12$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Simplex Table 1: Initial Solution

Basic		x_1	x_2	x_3	S_1	S_1	S_3	b_i	b_i/a_{ij}
S_1	0	1	2	2	1	0	0	8	4
S_2	0	3	2	6*	0	1	0	12	2(key row)
S_3	0	2	3	4	0	0	1	12	3
C_j		3	2	5	0	0	0		
Z_j		0	0	0	0	0	0		
$C_j - Z_j$		3	2	5	0	0	0		
				Key column					

Simplex Table 2: Non-optimal Solution

Basic		x_1	x_2	x_3	S_1	S_1	S_3	b_i	b_i/a_{ij}
S_1	0	0	4/3	0	1	-1/3	0	4	-
x_3	5	½*	1/3	1	0	1/6	0	2	4(key row)
S_3	0	0	5/3	0	0	-2/3	1	4	-
C_j		3	2	5	0	0	0		
Z_j		5/2	5/3	5	4	5/6	4	10	
C_j - Z_j		½	1/3	0	0	-5/6	0		
		Key column							

Simplex Table 3: Optimal Solution

Basic		x_1	x_2	x_3	S_1	S_1	S_3	b_i	b_i/a_{ij}
S_1	0	0	4/3	0	1	-1/3	0	4	3
X_1	3	1	2/3	2	0	1/6	0	4	6
S_3	0	0	5/3*	0	0	-2/3	1	4	12/5(key row)
C_j		3	2	5	0	0	0		
Z_j		3	2	6	4	1	0	12	
C_j - Z_j		0	0	-1	0	-1	0		

The solution contained in Table 3 is optimal with $x_1 = 4$, $x_2 = x_3 = 0$ and Z = 12. However, it is not unique since x_2, a non-basic variable, has $C_j - Z_j$ equal to zero. The problem, thus, has an alternate optimal solution. To obtain this, we revise the solution in Table 3with x_2 as the entering variable. It is given in Simplex Table 4.

Simplex Table 3: Alternate Optimal Solution

Basic		x_1	x_2	x_3	S_1	S_1	S_3	b_i
S_1	0	0	0	0	1	1/5	-4/5	4/5
x_1	3	1	0	2	0	3/5	-2/5	12/5
x_2	2	0	1	0	0	-2/5	3/5	12/5
C_j		3	2	5	0	0	0	
Z_j		12/5	12/5	0	4/5	0	0	12
C_j - Z_j		0	0	-1	0	-1	0	

2.3. Duality in Linear Programming.

For every linear programming problem there is another linear programming problem which is related to it and which is obtained from it. First problem is called primal and second is called its dual.

2.3.1. Rules for obtaining dual from primal :-

1) Co-efficients of variables in objective function of primal become constraint values in the dual and constraint values in the primal becomes coefficients of variables in the objective function.

2) If he primal is of maximization type, dual is of minimisation type and if primal is of minimisation dual is of maximisation.
3) Co-efficient of first column of constraints of primal because co-efficient of first row of dual, second column becomes second row and so on.
4) Direction of constraint in equations is also changed. It is primal they are of $\leq$ type, in dual they will be $\geq$ type.

Besides these, the following things should also be kept in mind :

i. All the variables in the dual must be non-negative.
ii. If the dual is of minimisation objective, all the constraints must be of £ type and if it is of minimisation, all the constraints must be of $\geq$ type. In any dual, we can't have mixed constraints.

Mathematically, change from primal to dual can be shown, with the help of an example.

2.3.2. Example. For the LPP given below, write the dual.

Maximise $Z = 40x_1 + 35x_2$

Subject to $2x_1 + 3x_2 \leq 60$

$4x_1 + 3x_2 \leq 96$

$x_1, x_2 \geq 0$

Solution. In accordance with above, its dual shall be

Minimise $G = 60y_1 + 96y_2,$

Subject to $2y_1 + 4y_2 \geq 40$

$3y_1 + 3y_2 \geq 35$

$y_1, y_2 \geq 0$

2.3.3. Obtaining Dual of LPP with Mixed Restrictions

Sometimes a given LPP has mixed restrictions so that the inequalities given are not all in the right direction. In such a case, we should convert the inequalities in the wrong direction into those in the right direction. Similarly, if an equation is given in respect of a certain constraint, it should also be converted into inequality. To understand fully, consider the following examples.

2.3.4. Example. Write the dual of the following LPP.

Minimise $Z = 10x_1 + 20x_2$

Subject to $3x_1 + 2x_2 \geq 18$

$x_1 + 3x_2 \geq 8$

$2x_1 - x_2 \leq 6$

$x_1, x_2 \geq 0$

Solution. Here, the first two inequalities are in the right direction (being $\geq$ type with a minimisation type of objective function) while the third one is not. Multiplying both sides by -1, this can be written as $-2x_1 + x_2 \geq 6$. Now, we can write the primal and dual as follows :

	Primal		Dual
Minimise	$Z = 10x_1 + 20x_2$	Maximise	$G = 18y_1 + 8y_2 - 6y_3$
Subject to		Subject to	
	$3x_1 + 2x_2 \geq 18$		$3y_1 + y_2 - 2y_3 \leq 10$
	$x_1 + 3x_2 \geq 8$		$2y_1 + 3y_2 + y_3 \leq 20$
	$-2x_1 + x_2 \geq -6$		
	$x_1, x_2 \geq 0$		$y_1, y_2, y_3 \geq 0$

2.3.5. Example. Obtain the dual of the LPP given here :

Maximise $Z = 8x_1 + 10x_2 + 5x_3$

Subject to

$x_1 - x_3 \leq 4$

$2x_1 + 4x_2 \leq 12$

$x_1 + x_2 + x_3 \geq 2$

$3x_1 + 2x_2 - x_3 = 8$

$x_1, x_2, x_3 \geq 0$

Solution. We shall first consider the constraints.

Constraints 1 and 2 ; Since they are both of the type $\leq$, we do not need to modify them.

Constraint 3 : This is of type $\geq$. Therefore, we can convert it into $\leq$ type by multiplying both sides by -1 to become $-x_1 - x_2 - x_3 \leq -2$.

Constraint 4 : It is in the form of an equation. An equation, mathematically, can be

represented by a part of inequalities: one of $\leq$ type and the other of $\geq$ type. The given constraint can be expressed as

$$3x_1 + 2x_2 - x_3 \leq 8$$

$$3x_1 + 2x_2 - x_3 \geq 8$$

The second of these can again be converted into type $\leq$ by multiplying by -1 on both sides. Thus it can be written as $-3x_1 - 2x_2 + x_3 \leq -8$.

Now we can write the primal and the dual as follows :

Primal			Dual
Maximise	$Z = 8x_1 + 10x_2 + 5x_3$	Minimise	$G = 4y_1 + 12y_2 - 2y_3 + 8y_4 - 8y_5$

Subject to

$x_1 - x_3 \le 4$

$2x_1 + 4x_2 \le 12$

$-x_1 - x_2 - x_3 \le -2$

$3x_1 + 2x_2 + x_3 \le -8$

$-3x_1 - 2x_2 + x_3 \le -8$

$x_1, x_2, x_3 \ge 0$

Subject to

$y_1 + 2y_2 - y_3 + 3y_4 - 3y_5 \ge 8$

$4y_2 - y_3 + 2y_4 - 2y_5 \ge 10$

$-y_1 - y_3 - y_4 + y_5 \ge 5$

$y_1, y_2, y_3, y_4, y_5 \ge 0$

One point needs mention here. We know that corresponding to an n-variable, m-constraint primal problem, there would be m-variable, n-constraint dual problem. For this example involving three variables and four constraints, the dual should have four variables and three constraints. But we observe that the dual that we have obtained contains five variables. The seeming inconsistency can be resolved by expressing $(y_4 - y_5) = y_6$, a variable unrestricted in sign. Thus, although, y_4 and y_5 are both non-negative, their difference could be greater than, less than, or equal to zero. The dual can be rewritten as follows :

Minimse $\quad G = 4y_1 + 12y_2 - 2y_3 + 8y_6$

Subject to

$y_1 + 2y_2 - y_3 + 3y_6 \ge 8$

$4y_2 - y_3 + 2y_6 \ge 10$

$-y_1 - y_3 - y_6 \ge 5$

$y_1, y_2, y_3 \ge 0$, y_6 unrestricted in sign

Thus, whenever a constraint in the primal involves an equality sign, its corresponding dual variable shall be unrestricted in sign. Similarly, an unrestricted variable in the primal would imply that the corresponding constraint shall bear the = sign.

2.3.6. Example. Obtain the dual of the following LPP :

Maximise $\quad Z = 3x_1 + 5x_2 + 7x_3$

Subject to $\quad x_1 + x_2 + 3x_3 \le 10$

$4x_1 - x_2 + 2x_3 \ge 15$

$x_1, x_2 \ge 0$, x_3 unrestricted in sign

Solution. First of all, we should convert the second restriction into the type $\le$. This results in $-4x_1 + x_2 - 2x_3 \le -15$.

Next, we replace the variable x_3 by the difference of two non-negative variables, say, x_4 and x_5. This yields the primal problem corresponding to which dual can be written, as shown against it.

Primal	Dual
Maximise $Z = 3x_1 + 5x_2 + 7x_4 - 7x_5$	Minimise $G = 10y_1 - 15y_2$
Subject to	Subject to
$x_1 + x_2 + 3x_4 - 3x_5 \leq 10$	$y_1 - 4y_2 \geq 3$
$-4x_1 + x_2 - 2x_4 + 2x_5 \leq -15$	$y_1 + y_2 \geq 5$
$x_1, x_2, x_4, x_5 \geq 0$	$3y_1 - 2y_2 \geq 7$
	$-3y_1 + 2y_2 \geq -7$
	$y_1, y_2 \geq 0$

The fourth constraint of the dual can be expressed as $3y_1 - 2y_2 \leq 7$. Now, combining the third and the fourth constraints, we get $3y_1 - 2y_2 = 7$. The dual can be expressed as follows :

Minimise $G = 10y_1 - 15y_2$

Subject to

$y_1 - 4y_2 \geq 3$

$y_1 + y_2 \geq 5$

$3y_1 - 2y_2 = 7$

$y_1, y_2 \geq 0$

The symmetrical relationship between the primal and dual problems, assuming the primal to be a 'maximisation' problem is depicted in the Chart.

Primal	Dual
Maximization	Minimisation
No. of variables	No. of constraints
No. of constraints	No. of variables
$\leq$ type constraint	Non-negative variable
= type constraint	Unrestricted variable
Unrestricted variable	= type constraint
Objective function coefficient for j^{th} variable	RHS constant for the j^{th} constraint
RHS constant for j^{th} constraint	Objective function coefficient for j^{th} variable
Coefficient (a_{ij}) for j^{th} variable in i^{th} constraint	Coefficient (a_{ij}) for i^{th} variable in j^{th} constraint

Comparing the Optimal Solutions of the Primal and Dual

Since the dual of a given primal problem is derived from and related to it, it is natural to except that the (optimal) solutions to the two problems shall be related to each other in the same way. To understand this, let us consider the following primal and dual problems again and compare their optimal solutions.

Primal	Dual
Maximise $Z = 40x_1 + 35x_2$	Minimise $G = 60y_1 + 96y_2$
Subject to	Subject to
$2x_1 + 3x_2 \leq 60$	$2y_1 + 4y_2 \geq 40$
$4x_1 + 3x_2 \leq 96$	$3y_1 + 3y_2 \geq 35$
$x_1, x_2 \geq 0$	$y_1, y_2 \geq 0$

The simplex table containing optimal solution to the primal problem is reproduced.

Simplex Table 1: Initial Solution

Basic		y_1	y_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
A_1	M	2	4*	-1	0	1	0	40	10(key row)
A_2	M	3	3	0	-1	0	1	35	35/3
C_j		60	96	0	0	M	M		
Z_j		5M	7M	-M	-M	M	M		
$C_j - Z_j$		60-5M	96-7M	M	M	0	0		

Simplex Table 2: Non-optimal Solution

Basic		y_1	y_2	S_1	S_2	A_1	A_2	b_i	b_i/a_{ij}
y_2	96	1/2	1	-1/4	0	1/4	0	10	20
A_2	M	3/2*	0	3/4	-1	-3/4	1	5	10/3 (key row)
C_j		60	96	0	0	M	M		
Z_j		48+3M/2	96	-24+3M/4	-M	24-3M/4	M		
$C_j - Z_j$		12-3M/2	0	24-3M/4	M	7M/4-24	0		

Simplex Table 2: Non-optimal Solution

Basic		y_1	y_2	S_1	S_2	A_1	A_2	b_i
y_2	96	0	1	-1/2	1/3	½	0	25/3
y_1	60	1	0	1/2	-2/3	-1/2	1	10/3
C_j		60	96	0	0	M	M	
Z_j		60	96	-18	-8	18	8	
$C_j - Z_j$		0	0	18	8	M-18	M-8	

Before comparing the solutions, it may be noted that there is a correspondence between variables of the primal and the dual problems. The structural variable x_1 in the primal, corresponds to the surplus variable S_1 in the dual, while the variable x_2 corresponds to S_2, the other surplus variable in the dual. In a similar way, the structural variables y_1 and y_2 in the dual correspond to the slack variables S_1 and S_2 respectively of the primal.

A comparison of the optimal solutions to the primal and the dual, and some observations follow.

a) The objective function values of both the problems are the same. This with $x_1 = 18$ and $x_2 = 8$, Z equals 40*18 + 35* 8 = 1000. Similarly, with $y_1 = 10/3$ and $y_2 = 25/3$, the value of G would be 60 *10/3 + 96 *25/3 = 1000.

b) The numerical value of each of the variables in the optimal solution to the primal is equal to the value of its corresponding variable in the dual contained in the C_j - Z_j row. Thus, in the primal problem, $x_1 = 18$ and $x_2 =8$, whereas in the dual $S_1 = 18$ and $S_2 = 8$ (in the C_j - Z_j row).

Similarly, the numerical value of each of the variables in the optimal solution to the dual is equal to the value of its corresponding variable in the primal, as contained in the C_j - Z_j row of it. Thus, $y_1 = 10/3$ and $y_2 = 25/3$ in the dual, and $S_1 = 10/3$ and $S_2 = 25/3$ (note that we consider only the absolute values) in the primal. Of course, we do not consider artificial variables because they do not correspond to any variables in the primal, and are introduced for a specific, limited purpose only.

Clearly then, if feasible solutions exist for both the primal and the dual problems then both problems have optimal solutions of which objective function values are equal. A peripheral relationship between them is that if one problem has an unbounded solution, its dual has no feasible solution.

Further, the optimal solution to the dual can be read from the optimal solution of the primal, and vice versa. The primal and dual need not both be solved, therefore, to obtain the solution. This offers a big computational advantage in some situations. For instance, if the primal problem is a minimization one involving, say 3, variables and 7 constraints, its solution would pose a big problem because a large number of surplus and artificial variables would have to be introduced. The number of iterations required for obtaining the answer would also be large. On the counter, the dual, with 7 variables and 3 constraints can be solved comparatively much more easily.

2.4. Transportation Problems.

If a company manufactures one products in two or more factories and has two or more main go-downs from where the product can be supplied to the customers, then the company has to decide how much quantity of each factory should be transported to each of the godown so that total transportation cost is minimised. Though we have other method to solve this problem, yet linear programming can also help in solving the transportation problems.

For example, a company has three plants P_1, P_2, P_3, and three warehouses W_1, W_2 and W_3 . Now various entities and costs can be shown in the form of the following matrix.

To / From	W_1	W_2	W_3	Supply
P_1	x_{11} C_{11}	x_{12} C_{12}	x_{13} C_{13}	S_1
P_2	x_{21} C_{21}	x_{22} C_{22}	x_{23} C_{23}	S_2
P_3	x_{31} C_{31}	x_{32} C_{32}	x_{33} C_{33}	S_3
Demand	D_1	D_2	D_3	

It is assumed that total supply = total demand.

In the above matrix c_{ij} represents transportation cost /unit from factory i to warehouse j and x_{ij} represents quantity (in units) transported from factory i to warehouse j.

Now

Objective function is

Minimise $Z = x_{11}\,c_{11} + x_{12}\,c_{12} + x_{13}\,c_{13} + x_{21}\,c_{21} + x_{22}\,c_{22} + x_{23}\,c_{23} + x_{31}\,c_{31} + x_{32}\,c_{32} + x_{33}\,c_{33}$

Subject to $x_{11} + x_{12} + x_{13} = S_1$

$x_{21} + x_{22} + x_{23} = S_2$ Supply constraints

$x_{31} + x_{32} + x_{33} = S_3$

$x_{11} + x_{21} + x_{31} = D_1$

$x_{12} + x_{22} + x_{32} = D_2$ Demand constraints.

$x_{13} + x_{23} + x_{33} = D_3$

$x_{ij} \geq 0$ for i = 1, 2, 3 and j = 1, 2, 3 .

As we can see that if we use simplex method to solve the above problem, having 9 decision variables and 6 constraints, it will be a long process and so this method is not generally used to solve transportation problems. So we shall confine ourselves to graphical method for solving these problems. In other words, we will have only two decision variables (say x and y).

2.4.1. Example. A company manufacturing a product has two plants P_1 and P_2 having weekly capacities 100 and 60 units respectively. The cars are transported to three godowns w_1, w_2 and w_3 whose weekly requirements are 70, 50 and 40 units respectively. The transportation costs (Rs./unit) are as given below:

P_1-w_1 = 5 , P_1-w_2 = 4 , P_1-w_3= 3, P_2-w_1= 4, P_2-w_2=2, P_3-w_2 =5.

Solve the above transportation problem so as to minimise total transportation costs.

Solution. First consider plant P_1. Let x and y be the units transported from P_1 to w_1 and w_2. Now we complete the matrix in the following form

To / From	w_1 cost/Qty. unit		w_2 cost/Qty. unit		w_3 cost/Qty. unit		Supply
P_1	5	x	4	y	3	(100-x-y)	100
P_2	4	(70-x)	2	(50-y)	5	(x+y-60)	60
Demand	70		50		40		160

Now total cost = 5x+4(70-x) + 4y + 2(50-y) +3(100-x-y) +5(x+y-60)

= 5x+280 -4x+4y+100-2y+300-3x-3y+5x+5y-300 = 3x+4y+380

So objective function is

Min. Z = 3x+4y+380

Subject to the constraints

(i) In first row $100-x-y \geq 0$ so $x+y \leq 100$

(ii) In 2nd row $70-x \geq 0$, $50-y \geq 0$ and $x+y-60 \geq 0$

So $x \leq 70$, $y \leq 50$ and $x+y \geq 60$

So we have 4 inequations

(i) $x+y \leq 100$ (ii) $x \leq 70$ (iii) $y \leq 50$ and (iv) $x+y \geq 60$.

Plotting these values on the graph, we get the following feasible region.

The feasible region lies in the area covered by the polygon ABCDE. We also know that optimal solution lies at one of the vertices. So now we find the values of x, y and z at these points.

Points	**x**	**y**	**Z = 3x+4y+380**
A	60	0	60*3+4*0+380 = 560
B	70	0	70*3+4*0+380 = 590
C	70	30	70*3+30*4+380 = 710
D	50	50	50*5+50*4+380 = 830
E	10	50	10*5+50*4+380 = 630

Since the minimum value of Z is Rs. 560 at A, so optimal values of x and y are x = 60, y = 0.

So the optimal transportation schedule is

From P_1, 60 units will be transported to w_1 and 40 units to w_3 .

From P_2, 10 units will be transported to w_1 and 50 units to w_2 .

2.5. Check Your Progress.

Solve the following linear programming using simplex method.

1. Maximise $Z = 7x_1 + 14x_2$

   Subject to the constraints

   $3x_1 + 2x_2 \leq 36$

   $x_1 + 4x_2 \leq 10$

   $x_1, x_2 \geq 0$

2. Maximise $Z = 20x_1 + 30x_2 + 5x_3$,

   Subject to

   $4x_1 + 3x_2 + x_3 \leq 40$

   $2x_1 + 5x_2 \leq 28$

   $8x_1 + 2x_2 \leq 36$

   and $x_1, x_2, x_3 \geq 0$

3. Maximise $Z = 10x_1 + 20x_2$,

   Subject to

   $2x_1 + 5x_2 \geq 50$

   $4x_1 + x_2 \leq 28$

   $x_1, x_2 \geq 0$

4. Minimise $Z = 6x_1 + 4x_2$

   Subject to

   $3x_1 + 0.5x_2 \geq 12$

   $2x_1 + x_2 \geq 16$

   $x_1, x_2 \geq 0$

5. Using two-phase Method, solve the following problem :

   Minimise $150x_1 + 150x_2 + 100x_3$,

   Subject to

   $2x_1 + 3x_2 + x_3 \geq 4$

   $3x_1 + 2x_2 + x_3 \geq 3$

   and $x_1, x_2, x_3 \geq 0$

6. Solve the following LPP :

Minimise $Z = 100x_1 + 80x_2 + 10x_3$,

Subject to

$100x_1 + 7x_2 + x_3 \geq 30$

$120x_1 + 10x_2 + x_3 \geq 40$

$70x_1 + 8x_2 + x_3 \geq 20$

and $x_1, x_2, x_3 \geq 0$

7. A pharmaceutical company produces two popular drugs A and B which are sold at the rate of Rs. 9.60 and Rs. 7.80, respectively. The main ingredients are x, y and z and they are

required in the following proportions :

Drugs	x%	y%	z%
A	50	30	20
B	30	30	40

The total available quantities (gm) of different ingredients are 1,600 in x, 1,400 in y and 1,200 in z. The costs (Rs) of x, y and z per gm are Rs. 8, Rs. 6 and Rs. 4, respectively. Estimate the most profitable quantities of A and B to produce, using simplex method. 8. A factory produces three different products viz. A, B and C, the profit (Rs) per unit of which are 3, 4 and 6, respectively. The products are processed in three operations viz. X, Y and Z and the time (hour) required in each operation for each unit is given below :

Operations	Products		
	A	B	C
X	4	1	6
Y	5	3	1
Z	1	2	3

The factory works 25 days in a month, at rate of 16 hours a day in two shifts. The effective working of all the processes is only 80% due to assignable causes like power cut and breakdown of machines. The factory has 3 machines in operation X, 2 machines in operation Y and one machine in operation Z. Find out the optimum product mix for the month.

9. A factory engaged in the manufacturing of pistons, rings and valves for which the profits per unit are Rs. 10, 6 and 4, respectively, wants to decide the most profitable mix. It takes one hour of preparatory work, ten hours of machining and two hours of packing and allied formalities for a piston. Corresponding requirements for rings and valves are 1, 4 and 2 , and 1, 5 and 6 hours, respectively. The total number of hours available for preparatory work, packing and allied

formalities are 100, 600 and 300, respectively. Determine the most profitable mix, assuming that what all produced can be sold.

10. A pharmaceutical company has 100 kg of material A, 180 kg of material B and 120 kg of material C available per month. They can use these materials to make three basic pharmaceutical products namely 5-10-5, 5-5-10 and 20-5-10, where the numbers in each case represent the percentage by weight of material A, material B and material C respectively, in each of the products and the balance represents inert ingredients. The cost of raw material is given below :

Ingredient	Cost per kg (Rs)
Material A	80
Material B	20
Material C	50
Inert ingredient	20

Selling price of these products is Rs. 40.50, Rs. 43 and Rs. 45 per kg respectively. There is a capacity restriction of the company for the product 5-10-5, that is, they cannot produce more than 30 kg per month. Formulate a linear programming model for maximising the monthly profit. Determine how much of each of the products should they produce in order to maximize their monthly profits.

11. The Clear-Vision Television Company manufactures models A, B and C which have profits Rs. 200, 300 and 500 per piece, respectively. According to the production license the maximum weekly production requirements are 20 for model A, 15 for B and 8 for C. The time required for manufacturing these sets is divided among following activities.

Activity	Time per piece (hours)			Total time available
	Model A	Model B	Model C	
Manufacturing	3	4	5	150
Assembling	4	5	5	200
Packaging	1	1	2	50

Formulate the production schedule as an LPP and calculate number of each model to be manufactured for yielding maximum profit.

12. A company produces two products, A and B. The sales volume of product A is at least 60 percent of the total sales of the two products. Both the products use the same raw material of which the daily availability is limited to 100 tonnes. Products A and B use this material at the rate of 2 tonnes per unit and 4 tonnes per unit, respectively. The sales price for the two products are Rs. 20 and Rs. 40 per unit.

    (a) Construct a linear programming formulation of the problem

    (b) Find the optimum solution by simplex method.

(c) Find an alternative optimum, if any.

Write the dual of the following linear programming problems.

13. Maximise $Z = 10y_1 + 8y_2 - 6y_3$

Subject to

$3y_1 + y_2 - 2y_3 \leq 10$

$-2y_1 + 3y_2 - y_3 \geq 12$

$y_1, y_2, y_3 \geq 0$

14. Maximise $Z = x_1 - x_2 + x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$2x_1 - x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

15. Maximise $Z = 3x_1 - 2x_2$

Subject to

$x_1 \leq 4$

$x_2 \leq 6$

$x_1 + x_2 \leq 5$

$-x_2 \leq -1$

$x_1, x_2 \geq 0$

16. Minimise $Z = 4x_1 + x_2$

Subject to

$3x_1 + x_2 = 2$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

17. Maximise $Z = 3x_1 + 4x_2 + 7x_2$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$4x_1 - x_2 - x_3 \geq 15$

$x_1 + x_2 + x_3 = 7$

$x_1, x_2 \geq 0$, x_3 unrestricted in sign.

18. Solve the following transportation problems :

Transportation cost (Rs./unit)

To From	W_1	W_2	W_3	Supply
F_1	6	3	2	100
F_2	4	2	3	50
Demand	60	50	40	150

19. A brick manufacturer has two depots A and B with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, q and R for 11000, 20000 and 15000 bricks respectively. The distance in kms. From these depots to the builder's location are given in the following matrix :

Transportation cost (Rs./unit)

To From	A	B
P	40	20
Q	20	60
R	30	40

How should the brick manufacturer fulfill the orders so that the total transportation costs are minimised ?

Answers.

1. $x_1 = 10, x_2 = 0, Z = 70$
2. $x_1 = 0, x_2 = 5.6, x_3 = 23.2, Z = 284$
3. $x_1 = 0, x_2 = 58, Z = 760$
4. $x_1 = 8, x_2 = 0, Z = 48$
5. $x_1 = 1/5, x_2 = 6/5, x_3 = 0, Z = 210$
6. $x_1 = 1/3, x_2 = 0, x_3 = 0, Z = 100/3$
7. A = 2000, B = 2000, Z = 10000 8 - A = 800/7, B = 0, C = 480/7, Z = 5280/7
8. Pistons = 100/3, Rings = 200/3, valves = nil, Z = 2200/3.

10. 30, 1185, 0, Z = Rs. 20625
11. A = 50/3, B = 15, C = 8, Z = 35500/3
12. (a) max. $Z = 20x_1 + 40x_2$ subject to $2x_1 + 4x_2 \leq 100$, $-8x_1 + 24x_2 \leq 0$, $x_1,x_2 \geq 0$

    (b) $x_1 = 30, x_2 = 10, Z = 1000$

    (c) $x_1 = 10, x_2 = 0, Z = 1000$
13. Min. $G = 10x_1 + 12x_2$ subject to $3x_1 + 2x_2 \geq 10$, $x_1 - 3x_2 \geq 8$, $2x_1 - x_2 \leq 6$, $x_1,x_2 \geq 0$
14. Min $G = 10y_1 + 2y_2 + 6y_3$ Subject to $y_1 + 2y_2 + 2y_3 \geq 1$, $y_1 - 2y_3 \geq -1$, $y_1 - y_2 + 3y_3 \geq 3$, $y_1,y_2,y_3 \geq 0$

15. Min. $G = 4y_1 + 6y_2 + 5y_3 - y_4$ Subject to $y_1 + y_3 \geq 3$, $y_2 + y_3 - y_4 \geq -2$, $y_1, y_2, y_3 \geq 0$

16. Max. $G = -2y_1 + 2y_2 + 6y_5$ Subject to $4y_3 - y_4 - 3y_5 \leq 4$, $3y_3 - 2y_4 - y_5 \leq 1$, $y_3, y_4, y_5 \geq 0$

17. Min. $G = 10y_1 - 15y_2 + 7y_3$ Subject to $y_1 - 4y_2 + y_3 \geq 3$, $y_1 + y_2 + y_3 \geq 4$, $3y_1 + y_2 + y_3 = 7$, $y_1, y_2 \geq 0$, y_3 unrestricted in sign.

18. From $F_1 \rightarrow$ 10 units to w_1, 50 units to w_2 and 40 units to w_3, From $F_2 \rightarrow$ 50 units to w_1 zero units to w_2 and w_3. Total transportation cost = Rs. 490.

19. From brick depot A - Zero to P, 20000 to Q and 10000 to R. From brick depot B - 15000 to P, zero to Q and 5000 to R. Total transportation cost = Rs. 1200.

2.6. Summary. In this chapter, we find the optimum solution systematically. As we have seen in graphical method, the vertices of the feasible region gives us feasible solutions. Simplex method helps us in finding the best solution from these feasible solutions.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan Chand and sons, Delhi.

3

Compound Interest

Structure

3.1. Introduction. In this chapter, we discuss about certain different types of interest rates and the concept of present value and amount of a sum.

3.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Different types of interest rates.

(ii) Present value.

(iii) Amount of a sum.

3.1.2. Keywords. Interest Rates, Compound Interest, Present Value.

3.2. Compound Interest.

Suppose a person takes a loan from a bank or from another person for a specified period of time. After this period, the amount he will return will be higher than the amount of loan taken. This additional amount will be paid by the borrower to the bank or second person for use of all loan given to him. This amount is called interest and the amount borrowed is called principal. Generally interest in expressed is percentage which is called rate of interest.

Interest is of two types :

1) Simple interest

2) Compound interest

If the lender is paid actual interest after every three months, six months or a year, it is called simple interest. But if this interest instead of being paid to the lender, is added to the principal and interest for next period is calculated on this new amount (principle + interest), it is called compound interest. In compound interest, the lender is paid full amount after completion of the period only once.

3.2.1. Example. Suppose Rs. 1000 is lent at 10% per annum for 2 years. Calculate simple interest and compound interest.

Solution. Simple Interest (S.I.)

S.I. for 1st year = $\frac{\mathbf{1000\times10\times1}}{\mathbf{100}}$ = Rs. 100

S.I. for 2nd year = $\frac{\mathbf{1000\times10\times1}}{\mathbf{100}}$ = Rs. 100

S.I. for 2 years = Rs. 100 + Rs. 100 = Rs. 200

Compund Interest (C.I.)

C.I. for 1st year = $\frac{1000\times10\times1}{100}$ = Rs. 100

After 1 year, this interest of Rs. 100 is not given to the lender but added to his principal.

So new principal = Rs. 1000 + Rs. 100 = Rs. 1100

Now C.I. for 2nd year = $\frac{1000\times10\times1}{100}$ = Rs. 110

So C.I. for 2 years = Rs. 100 + Rs. 110 = Rs. 210

3.2.2. Theorem. IF P is the principle, r % is the rate of interest per period and n is the number of periods, then

(i) Simple Interest (S.I.) = $\frac{\text{P.r.n}}{100}$ and

(ii) Compound Interest = $\text{P}\left(1+\frac{\text{r}}{100}\right)^{\text{n}} - \text{P}$

Proof. **(i)** S.I. of 1st period $= \frac{\mathbf{P.r.1}}{\mathbf{100}} = \frac{\mathbf{Pr}}{\mathbf{100}}$

S.I. of 2nd period $= \frac{\mathbf{P.r.1}}{\mathbf{100}} = \frac{\mathbf{Pr}}{\mathbf{100}}$

Continuing in this way.

S.I. of nth period $= \frac{\mathbf{P.r.1}}{\mathbf{100}} = \frac{\mathbf{Pr}}{\mathbf{100}}$

So total S.I. for n periods = S.I. for 1st period + S.I. for 2nd period +….+ S.I. for nth period

$$= \frac{Pr}{100} + \frac{Pr}{100} + \cdots n\ times$$

$$= \frac{\text{Pr}\,n}{100}$$

And amount A = P +S.I.

$$= P + \frac{\text{Pr}\,n}{100}$$

$$= P\left(1 + \frac{\text{r}\,n}{100}\right)$$

(ii) C.I for 1st period = $\frac{P.r.1}{100} = \frac{P.r}{100}$

Amount after 1st period = $P + \frac{Pr}{100} = P\left(1 + \frac{r}{100}\right)$

C.I. for 2nd period = $P\left(1 + \frac{r}{100}\right)\frac{r}{100}$

So amount after 2nd period = $P\left(1 + \frac{r}{100}\right) + P\left(1 + \frac{r}{100}\right)\frac{r}{100} = P\left(1 + \frac{r}{100}\right)\left[1 + \frac{r}{100}\right]$

$$= P\left(1 + \frac{r}{100}\right)^2$$

C.I. for 3rd period = $P\left(1 + \frac{r}{100}\right)^2 \frac{r}{100}$

So amount after 3rd period = $P\left(1 + \frac{r}{100}\right)^2 + P\left(1 + \frac{r}{100}\right)^2 \frac{r}{100} = P\left(1 + \frac{r}{100}\right)^2 \left[1 + \frac{r}{100}\right]$

$$= P\left(1 + \frac{r}{100}\right)^3$$

Amount after n periods = $P\left(1 + \frac{r}{100}\right)^n$

and Compound Interest

C.I. after n periods = $P\left(1 + \frac{r}{100}\right)^n - P$

$$= P\left[\left(1 + \frac{r}{100}\right)^n - 1\right]$$

Notes.

1) If n is not a whole number then it is divided into two parts – (i) a whole number part (k) and (ii) a fractional number (p) so n = k + p then

$$A = P\left(1 + \frac{r}{100}\right)^k \left(1 + \frac{Pr}{100}\right).$$

For example if n = 15 years 3 months, then n = 15 years +$\frac{1}{4}$ year and

$$A = P\left(1+\frac{r}{100}\right)^{15}\left(1+\frac{r}{4.100}\right)$$

2) Generally the unit of time period is in years. So the interest is compounded annually. In this case the above formula holds good. But if the interest is compounded monthly, quarterly or half yearly then calculations are changed as follows :

(i) Interest is compounded monthly

$$A = P\left(1+\frac{r}{12.100}\right)^{12n}$$

(ii) Interest is compound quarterly

$$A = P\left(1+\frac{r}{4.100}\right)^{4n}$$

(iii) Interest is compounded six monthly or half yearly

$$A = P\left(1+\frac{r}{2.100}\right)^{2n}$$

3) If the rate of interest (r%) changes every year i.e. r_1 in 1^{st} year, r_2 in 2^{nd} year,….., r_n in n^{th} year then

$$A = \boldsymbol{P}\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)\cdots\left(1+\frac{r_n}{100}\right)$$

3.2.3. Example. Find the compound interest on Rs. 50000 invested at the rate of 10% for 4 years.

Solution. P = Rs. 50000, r = 10%, n = 4 years

$$A = P\left(1+\frac{r}{100}\right)^{n}$$

$$= 5000\left(1+\frac{10}{100}\right)^{4}$$

$$= 5\ \times\frac{11}{10}\times\frac{11}{10}\times\frac{11}{10}\times\frac{11}{10}$$

$$=\ 5000\ \times 14641\ =\ Rs. 73205$$

$$C.I. =\ A-P\ \ =\ 73205-50000\ \ =\ Rs. 23205$$

3.2.4. Example. Ram deposits Rs. 31250 in a bank at a rate of 8% per annum for 3 years. How much amount will be get after 3 years. How much his earning will change, if interest is compounded half yearly.

Solution. (i) P = Rs. 31250, r = 8%, n = 3 years

$$A = 31250\left(1+\frac{8}{100}\right)^{3}$$

$$= 31250 \times \frac{27}{25} \times \frac{27}{25} \times \frac{27}{25} = \text{Rs. } 39366$$

(ii) If the rate is compounded half yearly, then

$$A = P\left(1 + \frac{r}{2.100}\right)^{2n}$$

$$= 31250\left(1 + \frac{8}{2.100}\right)^{2\times 3}$$

$$= 31250\left(1 + \frac{1}{25}\right)^{6} = Rs.\,39541.22$$

Change in earnings = $39541.22 - 39366 = Rs.\,175.22$

So if the interest rate is compounded half yearly, he will earn Rs. 175.22 more.

3.2.5. Example. Find the compound interest on a sum of Rs. 100000 at the rate 12% per annum for $2\frac{1}{2}$ years when the interest is compounded (i) annually, (ii) half yearly, (iii) quarterly (iv) monthly.

Solution. $P = Rs.\,100000,\ r = 12\,\%,\ n = 2\frac{1}{2}\ years.$

(i) Interest compounded annually

$$A = 100000\left(1 + \frac{12}{100}\right)^{2}\left(1 + \frac{12}{2.100}\right) = 100000 \times \left(\frac{28}{25}\right)^{2}\left(\frac{53}{50}\right)$$

$$Log\ A = Log\left[100000 \times \left(\frac{28}{25}\right)^{2}\left(\frac{53}{50}\right)\right]$$

$$= log\,100000 + 2[log\,28 - log\,25] + [log\,53 - log\,50]$$

$$= 5 + 2[1.4471 - 1.3979] + [1.7243 - 1.6990]$$

$$= 5 + 0.0984 + .0253$$

$$= 5.1237$$

$$A = AL[5.1237] = Rs.\,132953$$

So $\quad C.I. = 132953 - 100000 = Rs.\,32953$

(ii) Interest is compounded half yearly

$$A = 100000\left(1 + \frac{12}{2.100}\right)^{5} = 100000\left(1 + \frac{3}{50}\right)^{5} = 100000\left(\frac{53}{50}\right)^{5}$$

$$Log\ A = log\left[100000 \times \left(\frac{53}{50}\right)^{5}\right]$$

$$= log\,100000 + 5(log\,53 - log\,50) = 5 + .1265 = 5.1265$$

$$A = AL[5.1265] = Rs.\,133822$$

$C.I. = 133822 - 100000 = Rs. 33822$

(iii) Interest compounded quarterly

$$A = 100000\left(1 + \frac{12}{4.100}\right)^{\frac{5}{2}\times 4} = 100000\left(\frac{103}{100}\right)^{10}$$

$$Log\ A = log\left[100000 \times \left(\frac{103}{100}\right)^{10}\right]$$

$$= log\ 100000 + 10[log\ 103 - log\ 100]$$

$$= 5 + 0.1284 = 5.1284$$

$$A = AL[5.1284] = Rs. 134400$$

$$C.I. = 134400 - 100000 = Rs. 34400$$

(iv) Interest compounded monthly

$$A = 100000\left(1 + \frac{12}{12.100}\right)^{\frac{5}{2}\times 12} = 100000\left(\frac{101}{100}\right)^{30}$$

$$Log\ A = log\left[100000 \times \left(\frac{101}{100}\right)^{30}\right]$$

$$= log\ 100000 + 30\ [log\ 101 - log\ 100]$$

$$= 5 + 30\ [0.00432]$$

So $A = AL[5.1296] = Rs. 134785$

$C.I. = 134785 - 100000 = Rs. 34785$

3.2.6. Example. At what rate % will Rs. 32768 yield Rs. 26281 as compound interest in 5 years.

Solution. P = Rs. 32768

A = P + C.I.

= 32768 + 26281 = Rs. 59049

n = 5 years

Now

$$A = P\left(1 + \frac{r}{100}\right)^n$$

or

$$59049 = 32768\left(1 + \frac{r}{100}\right)^5$$

or

$$\frac{59049}{32768} = \left(1 + \frac{r}{100}\right)^5$$

or

$$\left(\frac{9}{8}\right)^5 = \left(1+\frac{r}{100}\right)^5$$

Therefore

$$1+\frac{r}{100} = \frac{9}{8}$$

$$r = \frac{100}{8} = 12.5\%$$

3.2.7. Example. At what rate % will a principal double itself in 6 years.

Solution. $$A = P\left(1+\frac{r}{100}\right)^n$$

$$2P = P\left(1+\frac{r}{100}\right)^6$$

or

$$\left(1+\frac{r}{100}\right)^6 = 2$$

Let $1+\frac{r}{100} = x$, then $x^6 = 2$. Taking logarithms of both sides

$$6\,log\,x = log\,2 = 0.3010$$

or $$log\,x = 0.0502$$

Therefore $$x = AL[0.0502] = 1.1225$$

So now $1+\frac{r}{100} = 1.1225$

or $$r = 12.25\,\%$$

3.2.8. Example. In how many years will Rs. 30000 becomes Rs. 43923 at 10% rate of interest.

Solution. $$A = P\left(1+\frac{r}{100}\right)^n$$

$$43923 = 30000\left(1+\frac{10}{100}\right)^n$$

or

$$\frac{43923}{30000} = \left(\frac{11}{10}\right)^n$$

or

$$\left(\frac{11}{10}\right)^4 = \left(\frac{11}{10}\right)^n$$

Therefore, n=4.

So in 4 years Rs. 30000 will become Rs. 43923 at 10% rate of interest.

3.2.9. Example. Sita invested equal amounts are at 8% simple interest and the other at 8% compound interest. If the latter earns Rs. 3466.40 more as interest after 5 years, find the total amount invested.

Solution. Let amount invested in each = P

So S.I. on P for 5 years at 8% = $\frac{P\times 8\times 5}{100} = \frac{2}{5}P$

and C.I. on P for 5 years at 8% $= \mathrm{P}\left(1+\frac{8}{100}\right)^5 - \mathrm{P}$

$$= \mathrm{P}\left[\left(\frac{27}{25}\right)^5 - 1\right]$$

$$\text{Difference} = \mathrm{P}\left[\left(\frac{27}{25}\right)^5 - 1\right] - \frac{2}{5}\mathrm{P}$$

$$= \mathrm{P}\left[\left(\frac{27}{25}\right)^5 - 1 - \frac{2}{5}\right]$$

$$= \mathrm{P}\left[\left(\frac{27}{25}\right)^5 - \frac{7}{5}\right] = \mathrm{P}\left[\frac{14348907}{9765625} - \frac{7}{5}\right]$$

$$= \frac{677032}{9765625}P$$

So now $\frac{677032}{9765625}P = 3466.40$

or $P = \frac{3466.40\times 9765625}{677032} = Rs. 50000$

So total amount invested = 50000 + 50000

= Rs. 100000

3.2.10. Example. A sum of money invested at C.I. becomes Rs. 28231.63 after 4 years and Rs. 3542.00 after 6 years. Find the principal and the rate of interest.

Solution. Let principal be P and rate of interest be r.

So $28231.63 = P\left(1+\frac{r}{100}\right)^4$ … (i)

And $33542.00 = P\left(1+\frac{r}{100}\right)^6$ … (ii)

Dividing (ii) by (i)

$$\frac{33542}{28231.63} = \left(1+\frac{r}{100}\right)^2$$

Put $1+\frac{r}{100} = x$, therefore

$$\frac{33542}{28231.63} = x^2$$

Taking logarithms of both sides

$$log\ 33542 - log\ 28231.63 = 2\ log\ x$$

$$4.52559 - 4.45073 = 2\ log\ x$$

or $$2log\ x = .07486$$

$$log\ x = 0.03743$$

or $$x = AL[0.03743] = 1.09$$

Therefore $1 + \frac{r}{100} = 1.09$

$$\frac{r}{100} = 1.09 - 1 = 0.09$$

or $r = 9\%$

Now substituting this value in equation (i)

$$28231.63 = P\left(1 + \frac{9}{100}\right)^4$$

or $$P = 28231.63\left(\frac{100}{109}\right)^4 = Rs. 20000$$

3.2.11. Example. The difference between S.I. and C.I. on a certain sum of money for 3 years at $8\frac{1}{2}\%$ rate of interest is Rs. 3566.26. Find the sum.

Solution. Let principal = Rs. P

$$S.I. = \frac{x \times 3 \times 17}{2 \times 100} = \frac{51}{100}P$$

$$C.I. = P\left(1 + \frac{17}{2 \times 100}\right)^3 - P$$

$$= P\left[\left(\frac{217}{200}\right)^3 - 1\right] = \frac{2218313}{8000000}P$$

Therefore $\frac{2218313}{8000000}P - \frac{51}{200}P = 3566.26$

or $\frac{2218313P - 2040000P}{8000000} = 3566.26$

or $\frac{178313}{8000000}P = 3566.26$

$$P = \frac{3566.26 \times 8000000}{178313}$$

$$= \frac{356626}{100} \times \frac{8000000}{178313} = Rs. 160000$$

3.2.12. Example. A person invests a part of Rs. 221000 at 10% C.I. for 5 years and remaining part for three years at the same rate. At time of maturity amount of both the investments is same. Find the sum deposited in each option.

Solution. Let principal in first option = P, r = 10% and n = 5 years

$$A = P\left(1 + \frac{10}{100}\right)^5 = P\left(\frac{11}{10}\right)^5$$

Sum invested in 2nd option = $(221000 - P)$

$$A = (221000 - P)\left(1 + \frac{10}{100}\right)^3 = (22100 - P)\left(\frac{11}{10}\right)^3$$

Now $\quad P\left(\frac{11}{10}\right)^5 = (22100 - P)\left(\frac{11}{10}\right)^3$

or $\quad \left(\frac{11}{10}\right)^2 = (22100 - P)$

or $\quad 121\,P = 22100000 - 100P$

or $\quad 221\,P = 22100000$

or $\quad P = \frac{22100000}{221} = Rs\ 100000$

So the sum invested in first option is Rs. 100000 and the sum invested in 2nd option is $(221000 - 100000)$ Rs. 121000

3.3. Continuos Compounding of Interest.

If the interest rate is compounded continuously, such that compounding frequency (δ) is infinitely large then

$$A = \lim_{\delta\to\infty} P\left[1 + \frac{r}{\delta.100}\right]^{n.\delta}$$

$$A = \lim_{\delta\to\infty} P\left[1 + \frac{r}{100\delta}\right]^{\left(\frac{100\delta}{r}\right)\left(\frac{nr}{100}\right)}$$

$$P = \left[\lim_{\frac{100\delta}{r}\to\infty}\left(1 + \frac{r}{100\delta}\right)^{\frac{100\delta}{r}}\right]^{\frac{nr}{100}}$$

$$= Pr^{\left(\frac{nr}{100}\right)} \quad \left[since\ \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m = e\right]$$

3.3.1. Example. Rs 8000 are invested at 6% per annum. Find the amount after 5 years if interest is compounded continuously.

Solution. $A = Pe^{\left(\frac{nr}{100}\right)}$

Now P = Rs. 8000, n = 3 years and r = 6%

$$A = Rs.(2.71825)^{\frac{3\times 6}{100}} = 8000 \times 1.197 = Rs. 9576.$$

3.3.2. Example. At what rate %, a sum will be doubled in 5 years if interest is compounded continuously.

Solution. $A = Pe^{\left(\frac{nr}{100}\right)}$

So $\quad 2P = P.e^{\left(\frac{5r}{100}\right)}$

or $\quad 2 = e^{\frac{r}{20}}$

Taking logarithms of both sides

$$\log 2 = \frac{r}{20}\log e$$

$$0.3010 = \frac{r}{20} \times 0.4343$$

or $$r = \frac{20\times .3010}{.4343} = 13.86\%$$

3.4. Effective Rate of Interest.

As we have seen in the example 3 that we get higher yields, if instead of annual compounding, interest is compounded monthly, quarterly or half yearly. So at the same rate of interest, we get higher interest as a result of increased compounding interest. Similarly if we want same interest in a given period, the effective rates will be higher if interest is compounded monthly, quarterly or half yearly instead of annually.

3.4.1. Example. A company offers 13% interest rate per annum on its debentures. What are the effective rates if interest is compounded (i) half yearly. (ii) quarterly (iii) monthly and (iv) continuously.

Solution. Let principal = Rs 100

Time = 1 year

Therefore , C.I. at 13% = $\frac{100\times 13\times 1}{100} = Rs. 13.$

(i) Interest compounded half yearly

$$A = P\left(1 + \frac{r}{2\times 100}\right)^{2n}$$

$$= 100\left(1 + \frac{13}{2\times 100}\right)^{2\times 1}$$

$$= 100 \times \frac{213}{200}\times\frac{213}{200} = 113.42$$

So effective rate of interest = $113.42 - 100 = 13.42\%$

(ii) Interest is compounded quarterly

$$A = P\left(1 + \frac{13}{4 \times 100}\right)^4$$

$$= 100 \times \left(\frac{413}{400}\right)^4 = 100 \times \frac{2.91 \times 10^{10}}{2.56 \times 10^{10}} = Rs. 113.67$$

So effective rate of interest = $113.67 - 100 = 13.67\%$

(iii) Interest is compounded monthly

$$A = P\left(1 + \frac{13}{12 \times 100}\right)^{12}$$

$$= 100 \times \left(\frac{1213}{1200}\right)^{12} = 100 \times \frac{10.147 \times 10^{10}}{8.916 \times 10^{10}} = Rs. 113.81$$

So effective rate or interest = $113.81 - 100 = 13.81\%$

(iv) Interest rate compounded continuously

$$A = Pe^{\left(\frac{nr}{100}\right)}$$

$$= 100 \times e^{\left(\frac{13 \times 1}{100}\right)}$$

$$= 100 \times (2.71828)^{.13}$$

$$= 100 \times 1.1388 = 113.88$$

So effective interest rate = $113.88 - 100 = 13.88\%$

So we can see that as frequency of compounding increases, effective interest rate also goes on increasing.

3.5. Check Your Progress.

1. Find the amount after 3 years if Rs. 16000 is invested at a rate of 10% per annum.
2. Find the compound interest earned on Rs. 5000 at a rate of 8% p.a. for 5 years.
3. Find the amount and compound interest on a sum of Rs. 80000 for $2\frac{1}{2}$ years at a rate of 6.5% p.a.
4. Find the difference in compound interest if interest is compounded (i) annually and (ii) half yearly on a sum of Rs. 20000 for 3 years at a rate of 6% p.a.
5. Find compound interest on Rs. 5000 at 8% p.a. compounded quarterly for nine months.
6. At what rate percent when annum will a sum double itself in 5 years.
7. At what rate percent per annum will Rs 20000 become Rs. 30000 in 3 years if the interest is compounded (i) half yearly and (ii) quarterly.
8. A person borrows certain amount of money at 3 % per annum simple interest and invests it at 5%

p.a. compound interest. After three years, he makes a profit of Rs 5410. Find the amount borrowed.

9. In how much time will a sum be doubled if the rate of interest is 10% per annum.
10. A certain sum of money becomes Rs. 5995.08 after 3 years at 6% p.a. find the principal.
11. The compound interest on a certain sum for 4 years at 8% rate is Rs. 404.89 more than simple interest on the same sum at the same rate and for the same time. Find the principal.
12. A sum of money amounts to Rs. 8988.8 in 2 years and to Rs. 10099.82 in 4 years at compound interest. Find the principal and the rate of interest.
13. Difference between C.I. and C.I. on a certain sum of money for 2 years at 5% p.a. is Rs 10. Find the sum.
14. A sum of Rs. 16896 is to be invested in two schemes one for 3 years and the other for 2 years. Rate of interest in both the schemes is 6.25% p.a. If the amount received at the maturity of the two schemes is same, find the sum invested in each scheme.
15. In how many years will a money treble itself at 8% if the interest is compounded continuously?
16. A company offers 12% rate of interest p.a. on its deposits. What is the effective rate of interest if it is compounded (i) six monthly (ii) quarterly (iii) monthly and (iv) continuously.
17. Which is better investment 8% compounded half yearly or 7.5% compounded quarterly.

Answers

1. Rs 21296	2. Rs. 2346.64	3. Rs. 93686.98 and Rs. 13686.98	
4.Rs. 60.73	5. Rs.307	6. 14.87 %	7. (i) 14%, (ii) 13.76%
8. Rs. 15912	9. 7.27 years	10. Rs. 5034	11. Rs. 10000
12. Rs.8000 & 6%	13. Rs. 4000	14. Rs. 8192 and Rs. 8704	
15. 8.53 years	16. (i) 12.36% (ii) 12.55% (iii) 12.68% and (iv) 12.75%		
17. Ist option			

3.6. Summary. In this chapter, we discussed compound interest, continuous compounding of interest, effective rate of interest and some examples related to these topics which help in understanding.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.

4

Annuities

Structure

4.1. Introduction. In this chapter, we discuss about various types of Types of annuities, their present value and amount of an annuity, including the case of continuous compounding.

4.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Annuity.

(ii) Present value of Annuity.

(iii) Deferred Annuity.

4.1.2. Keywords. Annuity, Present Value, Deferred Annuity.

4.2. Annuity.

Annuity is a series of equal payment made over equal interval of time periods.

For example, if a person deposits Rs. 2000 on first of every month for 2 years, it is an annuity. In this annuity amount of Rs. 2000 paid every month is called **instalment** of the annuity. Because the time difference between two instalments is one month, so the **payment** period of this annuity is one month. Besides this, since the time period between first and last payments is two years i.e. 24 months, so the **term** of the annuity is 24 months.

In annuity certain, number and amount of instalments is fixed and there is no change in then be causes of any contingency. For example instalment paid in recurring deposit in a bank, and for purchase of a plot of land are Annuities Certain.

In annuity contingent, instalments are paid till the happening of some specified event. For example, premium on an insurance policy is paid only as long as the policy holder is alive. In case of his/her then death before the maturity of the policy, further instalments are not paid.

In annuity perpetual, there is no time limit for payment of instalments, they are paid for ever. For example, the instalments of interest earned by endowment fund is a perpetual annuity as they are received regularly forever.

Besides this, if the payment of the instalments is made at the beginning of the corresponding period it is called a Annuity Due and if made at the end of the period, it is called Annuity Immediate. Annuity immediate is called ordinary annuity also. The total amount, to be received, after the maturity of the annuity is the sum of the accumulated values (principal + interest) of all the instalments paid.

Case I. When the annuity is annuity immediate

Let a and n be the amount and number of instalments of an annuity immediate. Further let r be the rate of interest per period. Since 1st instalment is paid at the end of first period, so it will earn on interest for (n-1) periods. Similarly 2nd instalment will earn interest for (n-2) periods and so an. Second last instalment will earn interest for 1 period only and last instalment will not earn any interest.

So total amount of the annuity

$$= a\left(1+\frac{r}{100}\right)^{n-1} + a\left(1+\frac{r}{100}\right)^{n-2} + \cdots + a\left(1+\frac{r}{100}\right) + a$$

$$= a\left[\left(1+\frac{r}{100}\right)^{n-1} + \left(1+\frac{r}{100}\right)^{n-2} + \cdots + \left(1+\frac{r}{100}\right) + 1\right]$$

$$= a[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1]$$

$$= a[1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1}]$$

Now this is a geometrical progression and so the sum is given by

$$\text{Amount} = a\left[\frac{1((1+i)^{n}-1)}{1+i-1}\right] = 0 \qquad \left\{since,\ \ S = \left[a\,\frac{(r^{n}-1)}{r-1}\right]\right\}$$

$$= a\left[\frac{1((1+i)^{n}-1)}{i}\right].$$

Case 2. When the annuity is annuity due.

In this case, first instalment is paid at the beginning of 1st period, 2nd instalment at the beginning of 2nd period and so on. So 1st instalment will earn interest for n period, 2nd instalment for (n-1) periods and so on. Last instalment will earn interest for one period only.

So Amount $= a\left(1+\frac{r}{100}\right)^n + a\left(1+\frac{r}{100}\right)^{n-1} + \cdots + a\left(1+\frac{r}{100}\right)$

$$= a[(1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)]$$

$$= a[(1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1} + (1+i)^n]$$

$$= a\left[\frac{(1+i)((1+i)^n - 1)}{1+i-1}\right]$$

$$= a\frac{(1+i)}{i}[(1+i)^n - 1]$$

or Amount$= a\left[\frac{(1+i)^n-1}{\frac{i}{1+i}}\right]$.

4.2.1. Example. A person deposits Rs. 2000 per month in a bank for 2 years. If bank pays compound interest at the rate of 8% p.a. find the amount he will receive if the annuity is (i) immediate and (ii) due.

Solution. $a = Rs. 2000, n = 12 \times 2 = 24$ months, $r = \frac{8}{12} = \frac{2}{3}\%$ monthly

(i) Annuity immediate

$A = a\left[\frac{(1+i)^n-1}{i}\right]$, where $i = \frac{2}{3} \times \frac{1}{100} = \frac{2}{300}$

$$\therefore \qquad A = 2000\frac{\left[\left(1+\frac{2}{300}\right)^{24} - 1\right]}{\frac{2}{300}}$$

$$= \frac{2000\times 300}{2}[1.1729 - 1] = 1000 \times 300 \times 0.1729$$

$$= Rs. 51870$$

(ii) Annuity due

$$A = a\left[\frac{(1+i)^n-1}{\frac{i}{1+i}}\right]$$

$$= 2000\frac{\left[\left(1+\frac{2}{300}\right)^{24}-1\right]}{\frac{2/300}{1+2/300}}$$

$$= \frac{2000\times(1.1729-1)}{\frac{2}{300}\times\frac{300}{302}} = 2000 \times 0.1729 \times \frac{302}{2}$$

$$= Rs. 52215.80$$

4.2.2. Example. Find the future value of an ordinary annuity of Rs. 4000 per year for 3 years 10% compound interest rate per annum.

Solution. Here $a = 4000, n = 3, i = \frac{10}{100} = 0.1$

So, $A = 4000\left[\frac{(1+0.1)^3-1}{0.1}\right]$

$$= \frac{4000}{0.1}(1.331-1) = 4000 \times 0.331 = Rs. 13240$$

4.2.3. Example. Find the future value of an annuity due of Rs. 5000 per year for 10 years at rate of 12 % p.a. the interest being compounded half yearly.

Solution. Here $a = 5000, n = 10 \times 2 = 20$ half years, $r = \frac{12}{2} = 6\%$ half yearly.

So $i = \frac{r}{100} = \frac{6}{100} = 0.06$

Now Amount $= 5000\left[\frac{(1+0.06)^{20}-1}{\frac{0.06}{1+0.06}}\right]$

$$= 5000[(1.06)^{20}-1] \times \frac{1.06}{0.06}$$

Let $x = (1.06)^{20}$

Therefore, $\log x = 20\ log 1.06$

$$= 20 \times 0.0253 = 0.5061$$

Therefore, $x = AL[0.5061] = 3.2071$

Now Amount $= 5000(3.2071-1) \times \frac{1.06}{0.06}$

$$= 5000 \times 2.2071 \times \frac{1.06}{0.06} = Rs. 194960.5$$

4.2.4. Example. Find the future amount of Rs. 50000 payable at the end of each quarter for 5 years at 10 % p.a. compounded quarterly.

Solution. $a = 40000, n = 5 \times 4 = 20$ quarters, $r = \frac{10}{4} = 2.5\%$ quarterly.

So $i = \frac{2.5}{100} = 0.025$

Now Amount $= a\left[\frac{(1+i)^n-1}{i}\right]$

$$= 5000\left[\frac{(1+0.025)^{20}-1}{0.025}\right]$$

Let $(1.025)^{20} = x$

Therefore, $\log x = 20\log(1.025) = 20 \times 0.0107 = 0.2145$

So $x = AL[0.2145] = 1.6386$

Thus Amount $= 50000 \times \left(\frac{1.6386-1}{0.025}\right)$

$$= 50000 \times \frac{0.6386}{0.025} = \text{Rs. } 127720$$

To find the instalment of given annuity when amount is given

4.2.5. Example. What instalment has a person to pay at the end of each year if he wants to get Rs. 5,00,000 after 10 years at 5% compound rate of interest per annum.

Solution. We know that

$$\text{Amount, } A = a\left[\frac{(1+i)^n-1}{i}\right]$$

Given $A = Rs. 500000, n = 10, r = 5$ so $i = 0.05$

Now $$500000 = a\left[\frac{(1+0.05)^5-1}{0.05}\right]$$

$$= a\left[\frac{(1.05)^{20}-1}{0.05}\right]$$

$$= a\left[\frac{1.6289-1}{0.05}\right] = \frac{0.6289}{0.05}a$$

Therefore, $$a = \frac{500000\times0.05}{0.6289} = Rs. 39751.95$$

4.2.6. Example. A company creates a sinking fund to provide for paying Rs. 1000000 debt maturing in 5 years. Find the amount of annual deposits at the end of each year if rate of interest is 18% compounded annually.

Solution. $A = 1000000, n = 5, r = 18\%$ so $i = \frac{18}{100} = 0.18$

$$A = a\left[\frac{(1+i)^n-1}{i}\right]$$

$$1000000 = a\left[\frac{(1+0.18)^5-1}{0.18}\right]$$

$$= a\left[\frac{(1.18)^5-1}{0.18}\right]$$

Let $$x = (1.18)^5$$

So $$\log x = 5log1.18 = 5 \times 0.07191 = 0.3595$$

$$x = AL[0.3595] = 2.2877$$

$$1000000 = a\left[\frac{2.5877-1}{0.18}\right]$$

or $$a = \frac{1000000\times0.18}{1.2877} = Rs. 151553.42$$

4.2.7. Example. A machine costs Rs. 1,50,000 and has a life of 10 years. If the scrap value of the machine is Rs. 5000, how much amount should be accumulated at the end of each year so that after 12 years a new machine could be purchased after 10 years at the same price. Annual compound rate of interest is 8 %.

Solution. Amount required after 10 years = $150000 - 5000 = 145000$

we are given $A = 145000, n = 12, r = 8\%, i = .08$

$$A = a\left[\frac{(1+i)^n-1}{i}\right]$$

$$145000 = a\left[\frac{(1+0.08)^{10}-1}{0.08}\right]$$

$$= a\left[\frac{2.1589-1}{0.08}\right]$$

Therefore, $a = \frac{145000 \times 0.08}{1.1589} = Rs. 10009.49$

4.2.8. Example. Find the minimum number of years for which an annuity of Rs. 2000 must sum in order to have at least total amount of Rs. 32000 at 5% compound rate of interest

Solution. $A = 32000, a = 2000, r = 5\%, i = .05$

$$32000 = 2000\left[\frac{(1+0.05)^n - 1}{0.05}\right]$$

or $\frac{32000 \times 0.05}{2000} = (1.05)^n - 1$

or $(1.05)^n = 1.8$

Taking logarithms of both sides

$$n. \log(1.05) = log1.8$$

$$n \times 0.0212 = 0.2553$$

$$n = \frac{0.2553}{0.0212} = 12.04$$

Therefore, the amount of annuity will take 13 years to exceed Rs. 32000 as total amount.

4.2.9. Example. What will be the instalment of an annuity having a total amount of Rs. 75000 for 12 years at 8% p.a., rate of interest compounded half yearly.

Solution. We are given that

$$A = 75000,\ n = 12 \times 2 = 24,\ r = \frac{8}{2} = 4\% \text{ and } i = 0.04$$

So $75000 = a\left[\frac{(1+0.04)^{24} - 1}{0.04}\right]$

or $75000 \times 0.04 = a[(1.04)^{24} - 1]$

Let $x = (1.04)^{24}$

$$logx = 24log1.04$$

$$= 24 \times 0.01703 = 0.4088$$

$$x = AL[0.4088] = 2.5633$$

So $75000 \times 0.04 = a(2.5633 - 1)$

$$a = \frac{3000}{1.5633} = Rs. 1919.02$$

Amount of an annuity when the interest is compounded continuously

In this case, the amount of the annuity is calculated by using the formula

$$A = a\int_0^n e^{it}dt \quad \text{where } i = \frac{r}{100}$$

4.2.10. Example. In an annuity, Rs. 5000 are deposited each year for 8 years. Find the amount if interest rate of 10% is compounded continuously.

Solution. We are given

$$a = 5000,\ n = 8,\ r = 10\ \% \text{ and } i = \frac{10}{100} = 0.10$$

So $$A = 5000\int_0^8 e^{0.1t}dt$$

$$= 5000\left[\frac{e^{0.1t}}{0.1}\right]_0^8$$

$$= \frac{5000}{0.1}[e^{0.8} - 1]$$

$$= 50000(2.71828^{0.8} - 1)$$

Let $$x = (2.71828)^{0.8}$$

$$logx = 0.8\ log2.71828$$

$$= 0.8 \times 04343 = 0.3474$$

$$x = AL[0.3474] = 2.2255$$

So $$A = 50000 \times (2.2255 - 1) = Rs. 61275$$

4.2.11. Example. A person wants to have Rs. 20000 in his recurring account at the end of 6 years. How much amount he should deposit each year if the rate of interest is 8 % p.a. compound continuously.

Solution. Here $A = 20000,\ n = 6,\ r = 8$ and $i = \frac{8}{100} = 0.08$

Now $$A = a\int_0^n e^{it}dt$$

$$20000 = a\int_0^6 e^{0.08t}dt$$

$$= a\left[\frac{e^{0.08t}}{0.08}\right]_0^6$$

$$= \frac{a}{0.08}[e^{0.48} - 1]$$

Let $$x = e^{0.48}$$

$$logx = 0.48\ log2 = 0.48 \times 0.4343 = 0.20846$$

$$x = AL[0.20486] = 1.6161$$

So $$2000 \times 0.08 = a[1.6161 - 1]$$

$$a = \frac{1600}{0.6161} = Rs. 2567$$

4.3. Present value of an annuity.

Present value of an annuity is equal to the total worth, at the time of beginning of the annuity, of all the future payments that are to be received. This value is equal to the sum of present values of all the instalments.

Let a be the amount of each instalment, n be the term (time periods) of the annuity and r % be the rate of interest per period. Further let V_1, V_2 … V_n be the present values of instalments paid in periods 1,2,…n respectively.

From our previous discussion, we know that the future value of (FV) of an annuity is given by

$$FV = a\left(1+\frac{r}{100}\right)^n$$

So if an instalment is paid at the end of period 1.

Then $$FV = a\left(1+\frac{r}{100}\right)^{n-1}$$

If we want to calculate present value (PV) of this instalment then it is calculated as

$$PV = \frac{a}{\left(1+\frac{r}{100}\right)}$$

So present value of an instalment is the amount of money today which is equivalent to the amount of that instalment, to be received after a specific period. In general, if an instalment, a, is paid in nth period and rate of interest is r % then

$$PV_n = \frac{a}{\left(1+\frac{r}{100}\right)^n} \quad or \quad \frac{a}{(1+i)^n}$$

Now we will find present value of both ordinary annuity and annuity due.

4.4. Ordinary Annuity or Annuity Immediate.

Present value of the annuity

$$\begin{aligned}
V &= V_1 + V_2 + V_3 + \cdots + V_n \\
&= \frac{a}{1+i} + \frac{a}{(1+i)^2} + \frac{a}{(1+i)^3} + \cdots + \frac{a}{(1+i)^n} \\
&= a\left[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^n}\right] \\
&= a\left[\left(\frac{1}{1+i}\right)\left(\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{1+i}}\right)\right] \\
&= a\left[\left(\frac{1}{1+i}\right)\left(\frac{(1+i)^n-1}{(1+i)^n}\right)\times\left(\frac{1+i}{1+i-1}\right)\right] \\
&= a\left[\frac{(1+i)^n-1}{(1+i)^n}\times\frac{1}{i}\right] = a\left[\frac{1-(1+i)^{-n}}{i}\right]
\end{aligned}$$

(ii) Annuity Due

In this type of annuity, each instalment is paid at the beginning of every period. So first instalment is paid at time zero, 2nd instalment at time 1 and so on last instalment is paid at period

(n-1) Hence PV of 1st instalment is equal to a, of 2nd instalment is $\frac{a}{1+\frac{r}{100}}$ of 3rd instalment is $\frac{a}{\left(1+\frac{r}{100}\right)^2}$ and

PV of last instalment is $\frac{a}{\left(1+\frac{r}{100}\right)^{n-1}}$.

So $$V = V_1 + V_2 + V_3 + \cdots + V_n$$

$$= a + \frac{a}{1+\frac{r}{100}} + \frac{a}{\left(1+\frac{r}{100}\right)^2} + \cdots + \frac{a}{\left(1+\frac{r}{100}\right)^{n-1}}$$

$$= a\left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}}\right]$$

$$= a\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{1+i}}\right]$$

$$= a(1+i)\left[\frac{1-(1+i)^{-n}}{\frac{1+i-1}{1+i}}\right]$$

$$= a(1+i)\left[\frac{1-(1+i)^{-n}}{i}\right] = a\left[\frac{1-(1+i)^{-n}}{\frac{i}{1+i}}\right].$$

4.4.1. Example. Find the present value of an ordinary annuity of Rs. 1500 per year for 5 years at 8% rate of interest.

Solution. $$V = a\left[\frac{1-(1+i)^{-n}}{\frac{i}{1+i}}\right]$$

Here $a = 1500,\ n = 5,\ r = 8$ and thus $i = \frac{8}{100} = 0.08$

So $$V = 1500\left[\frac{1-(1+0.08)^{-5}}{0.08}\right]$$

$$= \frac{1500}{0.08}[1-(1.08)^{-5}]$$

Let $$x = (1.08)^{-5}$$

$$log x = -5\ log 1.08$$

$$= -5 \times 0.0334$$

$$= (-0.1670 + 1) - 1$$

[Since, mantissa can never be negative so making the value positive, we add and substract 1 to the negative value]

$$= \bar{1}.8330$$

So $V = \frac{1500}{0.08}[1 - 0.6806]$

$= \frac{1500 \times 100}{8} \times 0.3194 = Rs. 5988.75$

4.4.2. Example. Find the present value of an annuity due of Rs. 800 per year for 10 years at a rate of 4 % p.a.

Solution. We are given

$a = 800,\ n = 10,\ r = 4$ % and so $i = \frac{4}{100} = 0.04$

Now $V = a\left[\frac{1-(1+i)^{-n}}{(1+i)}\right] = a[1-(1+i)^{-n}]\left(\frac{1+i}{i}\right)$

$= 800[1 - (1 + 0.04)^{-10})]\left(\frac{1.04}{0.04}\right)$

Let $x = (1.04)^{-10}$

$\log x = -10 \log(1.04)$

$= -10 \times 0.01703$

$= -0.1703 + 1 - 1 = \bar{1}.8297$

$x = AL[\bar{1}.8297] = 0.6756$

So now $V = 800[1 - 0.6756]\left(\frac{1.04}{0.04}\right)$

$= \frac{800 \times 0.3244 \times 1.04}{0.04} = Rs. 6747.52$

4.4.3. Example. A dealer sells a scooter to a customer on the condition that he will pay Rs 10000 in cash and balance to be paid in 36 month end instalment of Rs 400. If rate of interest is 12 % p.a. find the cash price of the scooter.

Solution. We are given

$a = 400,\ n = 36,\ r = 1$ % per month and so $i = \frac{1}{100} = 0.01$

Now $V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$

$= 400\left[\frac{1-(1+0.01)^{-36}}{0.01}\right]$

$= \frac{400}{0.01}[1 - (1.01)^{-36}]$

Let $x = (1.01)^{-36}$

$\log x = -36 \log(1.01)$

$= -36 \times 0.00432$

$= -0.15557 + 1 - 1 = \bar{1}.84443$

$x = AL[\bar{1}.84443] = 0.6989$

So now $V = 40000[1 - 0.6989]$

$= 40000 \times 0.3011 = Rs. 12044$

PV of 36 instalments = Rs. 12044

Cash paid = Rs. 10000

So cash price of the scooter = 12044 + 10000 = 22044

4.4.4. Example. A person takes a loan from a finance company for construction of a house, to be repayable in 120 monthly instalments of Rs. 1020 each. Find the present value of the instalments if the company charges interest @ 9 % p.a.

Solution. We are given

$a = 1020,\ n = 120,\ r = \frac{9}{12} = 0.75\%$ per month and $i = \frac{0.75}{100} = 0.0075$

Now $V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$

$= 1020\left[\frac{1-(1+0.0075)^{-120}}{0.0075}\right]$

$= \frac{1020}{0.0075}[1 - (1.0075)^{-120}]$

Let $x = (1.0075)^{-120}$

$\log x = -120 \log(1.0075)$

$= -120 \times 0.003245$

$= -0.3894 + 1 - 1 = \bar{1}.6106$

$x = AL[\bar{1}.6106] = 0.4079$

Hence $V = \frac{1020}{0.0075}[1 - 0.4079]$

$= \frac{1020}{0.0075} \times 0.5921 = Rs. 80525.64$

Type 2. To find amount of instalment when present value is given

4.4.5. Example. Find the amount of instalment on a loan of Rs 40000 to be payable in 10, at the end of year, equal instalments at a rate of 10 % interest per annum.

Solution. We are given

$V = 40000,\ n = 10,\ r = 10\ \%$ or $i = \frac{10}{100} = 0.1$

Now $V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$

$40000 = a\left[\frac{1-(1+0.1)^{-10}}{0.1}\right]$

or $$a = \frac{40000 \times 0.1}{1-(1.1)^{-10}}$$

Let $x = (1.1)^{-10}$

$$\log x = -10 \log(1.1)$$
$$= -10 \times 0.04139$$
$$= -0.4139 + 1 - 1 = \bar{1}.5861$$
$$x = AL[\bar{1}.5861] = 0.3856$$

Hence $$V = \frac{4000}{1-0.3856}$$
$$= \frac{4000}{0.6144} = Rs. 6510.42$$

4.4.6. Example. A person takes a loan of Rs. 600000 to be repaid in 60 equal end of month instalments at a rate of 8 % per annum. Find the amount of each instalment.

Solution. We are given

$$V = 600000,\ n = 60,\ r = \frac{8}{12} = \frac{2}{3}\% \text{ or } i = \frac{2}{300} = \frac{1}{150}$$

Now $$V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$$
$$600000 = a\left[\frac{1-\left(1+\frac{1}{150}\right)^{-60}}{\frac{1}{150}}\right]$$

or $$600000 \times \frac{1}{150} = a\left[1-\left(\frac{151}{150}\right)^{-60}\right]$$

Let $$x = \left(\frac{151}{150}\right)^{-60}$$
$$\log x = -60[\log(151) - \log(150)]$$
$$= -60(2.1790 - 2.17611)$$
$$= -60 \times 0.00289$$
$$= -0.17314 + 1 - 1 = \bar{1}.82686$$
$$x = AL[\bar{1}.82686] = 0.6712$$

Hence $$4000 = a[1 - 0.6712]$$
$$= a \times 0.3388$$
$$a = \frac{4000}{0.3388} = Rs. 11806.37$$

So monthly instalment is Rs 11806.37

Type 3. Interest is compounded continuously

Present value in this case is given by

$$V = a\int_0^n e^{-it}dt, \quad \text{where } i = \frac{r}{100}$$

4.4.7. Example. Find the present value of an annuity of Rs. 12000 per year for 4 years at a rate of 8 %. The interest is compounded continuously.

Solution. We are given

$$a = 12000,\ n = 4,\ r = 8\,\% \text{ or } i = \frac{8}{100} = 0.08$$

Now $$V = a\int_0^n e^{-it}dt$$

$$= a\left[\frac{e^{-it}}{-i}\right]_0^n$$

$$= -\frac{a}{i}\left[e^{-in} - 1\right]$$

So $$V = \frac{1200}{0.08}\left[(2.71828)^{-4\times 0.08} - 1\right]$$

Let $$x = (2.71828)^{-0.32}$$

$$\log x = -0.32 \log(2.71828)$$

$$= 0.32 \times 0.43429$$

$$= -0.1390 + 1 - 1$$

$$= \bar{1}.8610$$

$$x = AL[\bar{1}.8610] = 0.7261$$

So $$V = -150000 \times (0.7261 - 1)$$

$$= -150000 \times (-0.2739) = Rs.41085$$

4.5. Deferred Annuity.

Deferred annuity is an annuity in which payment of first instalment is made after lapse of some specified number of payment periods. This period is called deferment period. For example, payment of first instalment in case of educational loans and housing loans is paid after a deferment period of one to four years.

4.5.1. Amount of deferred annuity.

Let a be the instalment, n be the time periods, r % be the rate of interest and m the deferment period of a deferred annuity. Amount of this annuity is same as in case of other annuities. This amount is not affected by deferment period.

1. Annuity immediate

$$A = a\left[\frac{(1+i)^n - 1}{i}\right]$$

2. Annuity due

$$A = a\left[\frac{(1+i)^n - 1}{\frac{i}{1+i}}\right]$$

4.5.2. Present value of a deferred annuity.

Let $V_1, V_2 \ldots V_n$ be the present values of the 1st, 2nd, nth instalments respectively.

Case 1. Annuity immediate

Since m is the deferment period, so 1st instalment will be paid after (m+1) periods, 2nd after (m+2) periods and the last instalment is paid after (m+n) periods, so

$$V_1 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+1}}, \qquad V_2 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+2}}, \ \ldots \text{ and } \qquad V_n = \frac{a}{\left(1+\frac{r}{100}\right)^{m+n}}$$

$$V_1 = \frac{a}{(1+i)^{m+1}}, \qquad V_2 = \frac{a}{(1+i)^{m+2}} \quad \ldots.. V_n = \frac{a}{(1+i)^{m+n}}$$

Now the present value of the annuity

$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_n \\ &= \frac{a}{(1+i)^{m+1}} + \frac{a}{(1+i)^{m+2}} + \cdots + \frac{a}{(1+i)^{m+n}} \\ &= \frac{a}{(1+i)^m}\left[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n}\right] \\ &= \frac{a}{(1+i)^m}\left[\frac{\frac{1}{1+i}\left(1-\left(\frac{1}{1+i}\right)^n\right)}{1-\frac{1}{1+i}}\right] \\ &= \frac{a}{(1+i)^m}\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{i}\right] \\ &= a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right] \end{aligned}$$

Case II. Annuity Due

In this case, 1st instalment is paid after m periods, 2nd after (m+1) periods and so on. Last instalment will be paid after (m+n-1) periods.

So $$V_1 = \frac{a}{\left(1+\frac{r}{100}\right)^{m}}, \qquad V_2 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+1}}, \ \ldots \text{ and } \qquad V_n = \frac{a}{\left(1+\frac{r}{100}\right)^{m+n-1}}$$

$$V_1 = \frac{a}{(1+i)^{m}}, \quad V_2 = \frac{a}{(1+i)^{m+1}} \quad \ldots.. V_n = \frac{a}{(1+i)^{m+n-1}}$$

Now the present value of the annuity

$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_n \\ &= \frac{a}{(1+i)^{m}} + \frac{a}{(1+i)^{m+1}} + \cdots + \frac{a}{(1+i)^{m+n-1}} \\ &= \frac{a}{(1+i)^m}\left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}}\right] \end{aligned}$$

$$= \frac{a}{(1+i)^m}\left[\frac{\left(1-\left(\frac{1}{1+i}\right)^n\right)}{1-\frac{1}{1+i}}\right]$$

$$= a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{\frac{i}{1+i}}\right]$$

4.5.3. Example. Find the present value of a deferred annuity of Rs. 8000 per year for 8 years at 10 % p.a. rate, the first instalment to be paid at the end of 4 years.

Solution. We are given

$$a = 8000,\ n = 8,\ m = 4,\ r = 10\ \% \text{ so } i = \frac{10}{100} = 0.1$$

Now

$$V = a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$= 8000(1+0.1)^{-4}\left[\frac{1-(1+0.1)^{-8}}{0.1}\right] = 800(1.1)^{-4}\left[\frac{1-(1.1)^{-8}}{0.1}\right]$$

Let $x = (1.1)^{-4}$ and $y = (1.1)^{-8}$

$\log x = -4\log(1.1)$
$= -4 \times 0.04139$
$= -0.1656 + 1 - 1$
$= \bar{1}.8344$

$x = AL[\bar{1}.8344] = 0.6830$

$\log y = -8\log(1.1)$
$= -8 \times 0.04139$
$= -0.3312 + 1 - 1$
$= \bar{1}.6688$

$y = AL[\bar{1}.6688] = 0.4664$

So $$V = 8000 \times 0.6830\left[\frac{1-0.4664}{0.1}\right]$$

$$= 80000 \times 0.6830 \times 0.5336 = Rs. 29155.90$$

4.5.4. Example. A car is sold for Rs 75000 down and 30 half yearly instalments of Rs. 6000 each, the first to be paid after 4 years. Find the cash price of the car, if rate of interest is 12 % p.a. compounded half yearly.

Solution. We are given

$$a = 6000,\ n = 30,\ m = 3.5 \times 2 = 7,\ r = 6\ \% \text{ so } i = \frac{6}{100} = 0.06$$

Now

$$V = a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$= 6000(1+0.06)^{-7}\left[\frac{1-(1+0.06)^{-30}}{0.06}\right] = \frac{6000}{0.06}(1.06)^{-7}[1-(1.06)^{-30}]$$

Let $x = (1.06)^{-7}$ and $y = (1.06)^{-30}$

$\log x = -7\log(1.06)$

$\log y = -30\log(1.06)$

$= -7 \times 0.0253$ $\quad\quad\quad\quad\quad\quad$ $= -30 \times 0.0253$

$= -0.1771 + 1 - 1$ $\quad\quad\quad\quad\quad\quad$ $= -0.7592 + 1 - 1$

$= \bar{1}.8229$ $\quad\quad\quad\quad\quad\quad$ $= \bar{1}.2408$

$x = AL[\bar{1}.8229] = 0.6651$ $\quad\quad\quad\quad\quad\quad$ $y = AL[\bar{1}.2408] = 0.1741$

So $\quad V = 100000 \times 0.6651[1 - 0.174]$

$= 100000 \times 0.6651 \times 0.8259 = Rs. 54930$

Cash price = Cash payment + present value of future instalments

= 75000+54930

= Rs. 129930

4.6. Check Your Progress.

1. A person deposits Rs. 10000 at the end of each year for 5 years. Find the amount, he will receive after 5 years if rate of compound interest is 10% p.a.
2. Calculate the future value of an ordinary annuity of Rs. 8000 per annum for 12 years at 15% p.a. compounded annually.
3. A company has set up a sinking fund account to replace an old machine after 8 years. If deposits in this account Rs. 3000 at the end of each year and rate of compound interest is 5% p.a. find the cost of the machine.
4. To meet the expenses of her daughter a woman deposits Rs. 3000 every six months at rate of 10% per annum. Find the amount she will receive after 18 years.
5. A sinking fund is created by a company for redemption of debentures of Rs. 1000000 at the end of 25 years. How much funds should be provided at the end of each year if rate of interest is 4% compounded annually.
6. The parents of a child have decided to deposit same amount at every six months so that they receive an amount of Rs. 100000 after 10 year. The rate of interest is 5% p.a. compounded half yearly.
7. Which is a better investment - An annuity of Rs. 2000 each year for 10 year at a rate of 12 % compounded annually or an annuity of Rs. 2000 each year for 10 years at a rate of 11.75 % compounded half yearly.
8. Find the present value of an annuity due of Rs 4000 per annum for 10 years at a rate of 8 % per annum.
9. Find the present value of an ordinary annuity of Rs. 5625 per year for 6 year at rate of 9 % per annum.
10. Find the present values of an ordinary annuity of Rs. 5000 per six months for 12 years at rate of 4 % p.a. if the interest is compounded half yearly.
11. John buys a plot for Rs 3,00,000 for which he agrees to equal payments at the end of each year for 10 years . If the rate of interest is 10 % p.a. find the amount of each instalment.
12. Lalita buys a house by paying Rs 1,00,000 in cash immediately and promises to pay the balance amount in 15 equal annual instalments of Rs. 8000 each at 15 % compound interest rate. Find the cash price of the house.

13. Find the amount of instalment on a loan of Rs. 250000 to be paid in 20 equal annual instalments at a rate of 8 % per annum.
14. A persons buys a car for Rs. 2,50,000. He pays Rs. 1,00,000 in cash and promises to pay the balance amount in 10 annual equal instalments. If the rate of interest is 12 % per annum, find the instalment.
15. Find the present value of an annuity of Rs 11000 per year for 6 years at a rate of 11 % if the interest is compounded continuously.
16. Find the present value of a deferred ordinary annuity of 12000 per year for 10 years at a rate of 6 % p.a., the first instalment being paid after 3 years.

Answers.

1. Rs. 61050   2. Rs. 232008   3. Rs. 28647   4. Rs. 143754.48

5. Rs. 24081.9   6. Rs.3924.64

7. Amount 1st = Rs. 35098 and for 2nd = Rs. 34674. So first investment is better.

8. Rs. 28987.2   9. Rs. 25233.75   10. Rs. 94570   11. Rs. 1127.90

12. Rs. 146779   13. Rs. 25463.43   14. Rs. 26548.67   15. Rs. 53141

16. Rs. 74154.35

4.7. Summary. In this chapter, we discussed about annuities and its types and considered some examples to understand the topic.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.